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ABSTRACT

The purpose of this study was to describe and compare kindergarten and first-grade children's performance on addition and subtraction problems presented in two contexts: verbal (in which problem data were linked to physical referents such as objects or people and their actions), and abstract (in which no such links to physical situations occurred). Fifty kindergarteners and 54 first-graders were individually interviewed in mid-year to observe their solution strategies and errors on 12 abstract and 12 verbal addition and subtraction problems. The kindegarten problems contained sums and minuends less than 10. For first-graders, the sums and minuends ranged from 6 through 15. All problems were based on the open sentences a+b=?, a-b=?, and a+?=c. Upon completion of the problems, subjects in each grade were clustered according to the solution strategies they employed and according to the types of problems they could solve. Results indicated that verbal and abstract problems were of equal difficulty for subjects in both grades. Although kindergarteners used essentially the same strategies to solve verbal and abstract problems, first-graders exhibited less frequent use of concrete representation strategies on abstract than on verbal problems. Subjects in the two grades committed essentially the same types of errors, although the frequency of occurrence of most errors was lower at the first-grade level. At both grade levels a variety of individual differences were evident in the types of strategies subjects used and the types of problems they could solve. (Author/MP)

Technical Report No. 583

Kindergarten and First-Grade Children's Strategies for Solving Addition and Subtraction Problems in Abstract and Verbal Problem Contexts

by Glendon W. Blume

Technical Report No. 5[illegible]3

KINDERGARTEN AND FIRST-GRADE CHILDREN'S STRATEGIES FOR SOLVING ADDITION AND SUBTRACTION PROBLEMS IN ABSTRACT AND VERBAL PROBLEM CONTEXTS

by
Glendon W. Blume

Report from the Program on
Studies in Mathematics

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ABSTRACT

KINDERGARTEN AND FIRST-GRADE CHILDREN'S STRATEGIES FOR SOLVING ADDITION AND SUBTRACTION PROBLEMS IN ABSTRACT AND VERBAL PROBLEM CONTEXTS

Glendon Wilbur Blume

Under the supervision of Professor J. Fred Weaver

The purpose of this study was to describe and compare kindergarten and first-grade children's performance on addition and subtraction problems presented in verbal (word) problem and abstract contexts. Fifty kindergartners and fifty-four first-graders were individually interviewed in mid-year to observe their solution strategies and errors on twelve abstract and twelve verbal addition and subtraction problems.

The kindergarten problems contained sums and minuends less than ten, and for first-graders, sums and minuends ranged from six through fifteen. All problems were based on the open sentences a+b=_, a-b=_, and a+_=c. The verbal problems included action (Join) and static (Combine) addition, and action (Separate and Join/Change Unknown) subtraction problems. Subtraction problems included both small and large differences (e.g., 11-8=__ and 11-3=__). The abstract problems paralleled the

verbal problems and were presented in written number sentence mode to first-graders and oral mode (e.g., "Two and three are how many?") to kindergartners. All kindergarten subjects and half of the first-grade subjects had manipulatives available throughout the interviews.

At both grade levels there were no significant differences in the difficulty of verbal and abstract problems. However, subjects used strategies involving concrete representation (with manipulatives or fingers) less with abstract than verbal problems and guessing occurred more frequently on the abstract problems. At both grade levels and for problems in both contexts, the subjects' strategies mirrored the structure of the problems. The errors exhibited at the two grade levels were similar, but first-grade subjects experienced fewer difficulties in modeling the relationship or action in the problems.

Subjects in each grade were clustered according to the solution strategies they employed and according to the types of problems they could solve. The clusterings indicated that subjects who were homogeneous in terms of the types of problems they could solve were heterogeneous in terms of the solution strategies they employed.

The study indicates that abstract and verbal addition and subtraction problems are of equal difficulty for children at the kindergarten and first-grade levels. This suggests that verbal problems are a potential vehicle for initial instruction on the operations of addition and subtraction.

Chapter I

STATEMENT OF THE PROBLEM

A substantial portion of children's early elementary mathematics instruction focuses on the operations of addition and subtraction. The considerable instructional emphasis accorded to these operations is warranted by their mathematical importance. Simple addition and subtraction are encountered often in people's day-to-day activities. Computational skills with addition and subtraction are universally accepted as "basic skills" and invariably appear in lists of minimal competencies. In addition to their importance in everyday life, the operations of addition and subtraction are ubiquitous in all levels of mathematics and are often referred to as two of the "four fundamental operations." Thus, curriculum developers and teachers justifiably devote considerable attention to addition and subtraction in elementary school mathematics.

Woody (1931) and Ginsburg (1977a) contend that prior to formal schooling, children possess important concepts and skills concerning mathematics. In particular, within their natural environment, children develop informal ways of dealing with addition and subtraction. When designing effective instruction and appropriate instructional materials, it is necessary to take into account the informal strategies, procedures, or skills children bring to the instructional process. During the course of instruction, knowledge about children's ability to create physical models for addition and subtraction problems and information about their strategies and errors can help teachers to guide children toward more abstract and efficient strategies. After initial instruction has occurred, the types of strategies a child uses and the flexibility with which they are used can provide information about the child's general conceptual models of addition and subtraction. Hence, it is important for teachers and curriculum developers to have access to information about how students solve addition and subtraction problems before, during, and after initial instruction on these topics.

Instruction related to the operations of addition and subtraction can occur in a variety of contexts. The problems used can be verbal ("word" or "story") problems, i.e., ones in which the problem is written in prose and the problem data are embedded in a physical situation. "Frank had three stamps. His mother gave him 6 more stamps. How many stamps did Frank have altogether?" is an example of a verbal problem. Instruction can also entail abstract problems in which the problem data are presented without being related to any physical referents. "Six take away four is how many?" and "3+__=11" are examples of abstract problems. Problems presented in verbal and abstract contexts are quite different. Consequently, it is important for teachers and curriculum developers to know how children differ in their approaches to addition and subtraction in these contexts. Knowledge about these differences can contribute to decisions concerning how the addition and subtraction operations are best introduced in the school mathematics curriculum.

Purpose of the Study

The purpose of this study was to describe and compare kindergarten and first-grade children's performance on addition and subtraction tasks presented in verbal and abstract contexts. The study provided a detailed cross-sectional description of how children solve certain addition and subtraction problems. This description was done both for children who had received no formal instruction on addition and subtraction and for children who had received initial instruction on these operations. The study focused on the processes children use to solve addition and subtraction problems, i.e., their attempts to create models or representations of the numbers and relationships or actions described in the problems, and their solution strategies and errors. Such information on processes is necessary for understanding individual differences in the way children approach such problems and for determining the implications of those differences for instruction.

Previous studies have investigated kindergartners' and first-graders' performance on verbal and/or abstract addition and subtraction problems, but the focus of much of that research was on problem difficulty. Until recently, most studies generated difficulty indices and attempted to identify task and subject variables influencing difficulty rather than attempting to examine children's problem solving processes. Recent research has begun to focus more on the strategies children use to solve problems in either verbal or abstract contexts. The present study was designed to compare children's strategies for solving verbal and abstract addition and subtraction problems. This comparison is important because different studies have generated competing hypotheses about how children solve problems in the two contexts.

Questions Addressed by the Study

The questions of interest in the present study were concerned with kindergartners' and first-graders' ability to solve certain abstract and verbal addition and subtraction problems and the processes they use for solving them. The main questions fell into three categories: questions concerning differences in performance on abstract and verbal problems, questions dealing with differences between the strategies exhibited by children who have had no instruction on addition and subtraction and those of children who have experienced initial instruction, and questions concerning individual differences in young children's solution strategies and ability to solve certain addition and subtraction problems.

The first group of questions derived from concerns about the most appropriate context in which to introduce the addition and subtraction operations in school mathematics. A common assumption made by school mathematics programs and teachers is that verbal problems are difficult for children to solve and that children must master addition and subtraction facts before they can solve verbal problems. Carpenter, Hiebert, and Moser (1981) and Carpenter and Moser (1981) provide data indicating that this assumption may be false and argue that verbal problems are a viable vehicle for initial work with addition and subtraction. Documentation of differences in children's ability to solve addition and subtraction problems in the two contexts, verbal and abstract, is necessary for choosing among alternative approaches to initial instruction on these operations. Such choices also depend on differences in the processes children use to solve verbal and abstract problems.

The preceding considerations generated several questions. Are kindergarten and first-grade children better able to solve verbal or abstract problems? Do children use different strategies for solving problems in these two contexts? A reasonable hypothesis might be that cues from the physical situations make verbal problems easier for kindergartners to solve than abstract problems. Such differential difficulty might not be expected to occur with first-graders whose instruction on addition and subtraction has focused primarily on abstract problems. The cues inherent in verbal problems might, however, lead to different strategies being used for verbal and abstract problems. This raised several further questions. Do the solution strategies used by kindergarten and first-grade children mirror the structure of the problem to the same extent on verbal as on abstract problems? Do children use different strategies on verbal or abstract problems containing small numbers from those they use on problems in which the numbers are larger? The present study provided data pertinent to the preceding questions with the intent of presenting a detailed comparison of children's strategies for solving and ability to solve problems in two contexts which can be included in initial instruction relating to the operations of addition and subtraction.

Other questions of interest dealt with differences between the solution processes used by kindergartners and first-graders, i.e., children without formal instruction on addition and subtraction and children who have experienced initial instruction. Do kindergartners use a variety of procedures for solving verbal addition and subtraction problems? Do these solution strategies vary depending on the type of problem, as was the case with first-graders in Carpenter and Moser (1981)? It was hypothesized that kindergartners would use strategies which mirrored the content and structure of the problems but that their strategies would be less abstract than those used by first-graders. A related question is whether or not first-graders use solution strategies more interchangeably than kindergartners, i.e., whether first-graders use a variety of strategies on a given type of addition or subtraction problem. The present study addressed the preceding questions by providing cross-sectional data on solution strategies; similar data on children's errors were pertinent to questions concerning the mistakes and misconceptions exhibited by children at these grade levels. In particular, are kindergartners' errors both qualitatively and quantitatively different from those of first-graders? Do kindergartners misinterpret addition and subtraction problems and use the wrong operation in their solutions or are their errors predominantly procedural errors such as miscounting? By addressing the preceding questions the present study was able to compare a variety of aspects of the processes used by children with no formal instruction on addition and subtraction to those of children who had received instruction thereon.

A third group of questions was concerned with individual rather than group differences in children's approaches to addition and subtraction problems. To what extent do children differ in their capabilities for solving abstract and verbal addition and subtraction problems? Can individuals be clustered meaningfully according to the types of solution strategies they use on addition and subtraction problems? Such a description of individual differences is useful both for instructional design and for classroom diagnosis. The design of initial instruction on the operations of addition and subtraction can be guided by information about the variety and frequency of solution strategies used by kindergartners and first-graders, and knowledge of the types of errors exhibited by individuals can provide a basis for diagnosis of the misconceptions held by young children.

Description of the Study

The discussion which follows briefly describes the procedures chosen to address the questions presented in the preceding section. A detailed description of empirical procedures is given in Chapter III.

Tasks Used in the Study

The items used in the study consisted of a variety of simple addition and subtraction problems. These problems were presented in two contexts: verbal problems in which problem data were linked to physical referents such as objects or people and their actions, and abstract problems in which no such links to physical situations occurred. The abstract addition problems were of the form a+b=__ and were presented orally to kindergartners and in written number sentence format to the first-graders. The verbal addition problems involved either action or static relationships. "Bill had three books. His friend gave him eight more books. How many books did Bill have altogether?" is an example of an action problem, and "Jane has four red books. She also has seven green books. How many books does Jane have altogether?" is a problem involving a static relationship of parts to a whole.

The two types of abstract subtraction problems used were based on a-b=__ and a+__=c. Verbal subtraction problems paralleled the abstract ones and involved action, as in "Bill had eight books. He gave six of them to Martha. How many books did Bill have left?" or "Bill has two books. How many books does he have to put with them so he has eight books altogether?"

Administration of the Tasks

Twelve abstract and twelve verbal problems were separately administered to each subject in two individual interviews. Manipulative objects (cubes) were available to all kindergartners and half of the first-graders throughout the interviews. During the interviews the subjects solved the problems without assistance from the interviewer and were often asked to give retrospective accounts of their procedures upon completion of a problem. When necessary, additional probing questions were asked of the child to clarify solution processes that appeared ambiguous.

Although the present study gathered data on correct responses, its primary focus was on process-related aspects of children's solutions. Attempts to construct physical models of the problem data, solution strategies, and errors were observed and coded by the interviewer. A written record was also made of anecdotal data that were considered pertinent to accurate characterization of individuals' solution processes.

Scope of the Study

The intent of this section is to place the study within the context of mathematics education research in general ~~and research~~ on addition and subtraction in particular. This section will delimit the study in terms of its purpose and implications.

Romberg and DeVault (1967) identified components of the mathematics curriculum which serve as an organizational framework for research. The present study addressed two of these components--mathematics programs and the learner. It identified individual differences among learners' solution processes and their ability to solve addition and subtraction problems with the intent of contributing to the specification of appropriate content in that portion of the mathematics curriculum concerned with initial problem solving experiences in addition and subtraction.

Within mathematics education research a variety of research strategies can be employed. Romberg (1970) distinguished two types of research, evaluative and elemental research. The present study is classified as that type of elemental research in which the source of the questions is practice-based rather than theory-based.

One type of practice-based elemental research is relational research such as status studies, survey research, and correlational research. The present investigation was a status study, the intent of which was to describe the existing relationship of certain variables (problem presentation context, problem structure, number size, and grade level) to children's solution processes for addition and subtraction problems.

The scope of the present study can be delimited in several ways. As a status study the investigation was concerned with assessing solution processes used by children at two levels, kindergartners prior to instruction on the operations of addition and subtraction and first-graders after initial instruction. It was not concerned with testing the effects of alternative instructional programs. Although learning may have occurred during the interviews, an attempt was made to avoid any direct instruction during the interviewer/child interaction. Consequently, the study was not an instructional study focusing on intervention. The study incorporated tasks in both verbal and abstract contexts but problems in the two contexts were administered independently. It was not a study which attempted to investigate children's ability to represent verbal problems with number sentences, nor did it attempt to determine prerequisite relationships among the ability to solve verbal problems and the ability to construct a concrete model or write a number sentence to represent a verbal problem.

The preceding delimitations served to focus the study on two of four potential areas of research on verbal and abstract addition and subtraction problems. One area involves the relative difficulty of various verbal or abstract problems and includes studies investigating the influence of subject and task variables on the difficulty of abstract or verbal addition and subtraction problems. The second area involves the processes children use to solve verbal or abstract problems and is concerned with children's internal representation of addition and subtraction problems. This area of research attempts to describe the development of children's general conceptual models for addition and subtraction. A third area focuses on the logical relationship between abstract and verbal problems and the prerequisite relationships among the ability to solve a verbal problem, the ability to write a number sentence to model the verbal problem, and the ability to solve the number sentence. The fourth area deals with instruction and encompasses research concerned with the timing of the introduction of written symbolism for addition and subtraction and with methods of instruction that incorporate varying emphases on verbal and abstract problems. The present study addressed the first and second areas, comparing both difficulty and process-related aspects of performance on verbal and abstract addition and subtraction problems.

By focusing primarily on children's processes this study differed in emphasis from much of the previous research on addition and subtraction. The focus of the present study is consistent with Lindvall and Ibarra's (1980) suggestion for research aimed at "identifying the general nature of the understanding of number sentences held by those students who are successful in solving them" and identifying "what it is that pupils do when they arrive at correct answers to open sentences" (p. 60). They argue that studying the strategies children employ can be a useful first step to studying understanding of abstract addition and subtraction items. The purpose of the present study is also consistent with Vergnaud's (1979) contention that, although it is important to determine item difficulties, one should also study the complexity of procedures or strategies used on verbal problems.

Rationale and Significance of the Study

Since children enter school with some counting skills already established and since early addition and subtraction strategies are often counting-based (Carpenter & Moser, 1981; Ginsburg & Russell, Note 1), it is logical to investigate whether kindergartners can solve addition and subtraction problems. The research reviewed in Chapter II provides data indicating that some kindergarten children possess limited aspects of such ability. Although some information exists on kindergartners' performance on addition and subtraction, extant research provides no detailed description of the processes kindergartners use when attempting to solve addition and subtraction problems in different contexts. Carpenter and Moser (1981) described strategies used by first-graders on selected addition and subtraction verbal problems. The first-grade verbal problem interviews in the present study in part replicate the Carpenter and Moser work and incorporate a more detailed analysis of errors.

Comprehensive and systematic descriptions of children's modeling, strategies, and errors prior to formal instruction on addition and subtraction are necessary for one to describe individual differences in children's approaches to and ability to solve addition and subtraction problems. The diagnostic information resulting from a focus on processes can provide information concerning children's readiness for later instruction on addition and subtraction. Detailed accounts of children's processes and errors yield information about the misconceptions that occur prior to instruction and are helpful for both the design of initial instruction and later work with children who continue to have difficulty solving addition and subtraction problems.

An emphasis on children's strategies also allows information to be gathered which documents children's early problem solving procedures. The extent to which children "invent" strategies and their use of simple problem solving heuristics potentially shed some light on later difficulties with or capabilities for more complex mathematical problem solving.

No previous research has attempted to describe children's modeling procedures, strategies, and errors on a range of abstract problems. The present study serves to fill that void and to extend the Carpenter and Moser (1981) work with verbal problems by documenting processes used on corresponding abstract items.

A major contribution of the present study is the comparison of children's performance on addition and subtraction problems in abstract and verbal contexts. The extent to which children can solve either verbal or abstract items and whether or not competence on problems in one presentation context precedes competence with comparable problems in the other context is important for curriculum decisions concerning content sequencing, i.e., decisions about when and how to introduce the operations of addition and subtraction in the school mathematics curriculum. For example, in the absence of direct instruction thereon, if verbal problems were more difficult than abstract problems, one alternative would be to proceed by initially postponing instruction on verbal addition and subtraction problems. However, if verbal problem solving were itself deemed an important objective, one might nevertheless opt for instruction that utilized both contexts. Conversely, if competence with verbal problems appeared first, one might build initial experiences with addition and subtraction around verbal problems. In order to promote the development of children's strategies for solving addition and subtraction problems it is necessary to determine the strategies children use on abstract and verbal problems and to build upon children's naturally occurring or "invented" strategies.

The cross-sectional nature of the present study provides data that compare performance in kindergarten and grade one. Little is known about the shift from less to more abstract and efficient strategies across these two grade levels. The study, while not longitudinal in design, compares the types of errors, the stratgies used, and attention to problem structure exhibited in kindergarten and grade one, and contributes to the generation of hypotheses about the development of children's addition and subtraction problem solving processes.

The chapters that follow present a detailed description of the study. Chapter II reviews the related research which served as the background for the study. The specific questions addressed by the study and the empirical procedures chosen to gather data pertinent to those questions are given in Chapter III. Chapter IV presents the results of the study and the final chapter includes interpretation and discussion of the results.

Chapter II

REVIEW OF RESEARCH

The present study was designed to describe and compare kindergarten and first-grade children's solution processes for verbal and abstract addition and subtraction problems. Existing literature related to the study includes a considerable body of empirical research on addition and subtraction as well as theoretical analyses aimed at classifying addition and subtraction verbal problems. Since both the present investigation and previous research incorporated a variety of addition and subtraction problems, the initial section of this chapter reviews attempts to classify addition and subtraction problem types. It also provides a description of solution strategies for addition and subtraction problems that previous researchers have identified.

The second section of the review of related literature focuses on research on the difficulty of addition and subtraction problems and the solution processes used by children with three levels of instructional experience with addition and subtraction. Since the present study used kindergarten subjects who had received no formal instruction on addition and subtraction and first-grade subjects who had received initial instruction on these operations, there are natural distinctions among related studies according to the instructional background of the subjects. These are: a) studies using subjects prior to their initial instruction on addition and subtraction (preschoolers, kindergartners and some first-graders), b) studies with first-graders who had received initial instruction on these operations, and c) those studies that used older children. Furthermore, since problems in verbal and abstract contexts were used in the present study, this distinction serves as an appropriate means for subdividing the studies within each of the preceding three levels.

A majority of research on addition and subtraction has been concerned with determination of difficulty indices for various problem types and identification of task and subject variables that influence problem difficulty. Although the major purpose of the present investigation was not the determination of item difficulties, studies that focused on certain aspects of item difficulty are pertinent to the present investigation. These studies influenced the choice of item types appropriate for use with subjects within each age level and they offered data on one type of comparison of performance on verbal and abstract problems, i.e., comparison of difficulty.

A second focus of research on addition and subtraction has been on determining the solution processes children use when solving addition and subtraction problems. When interpreting research on the strategies children use to solve addition and subtraction problems it is necessary to keep in mind that this research has been derived from two very different paradigms. One approach is inferential in that it fits performance data to hypothesized behaviors or solution methods. The inferential paradigm includes the inference of strategies or solution methods from latency of correct responses or from an analysis of errors. The other approach is direct and attempts to ascertain the strategies used by children through direct observation and questioning. Within each of the three levels of subjects' instructional background, the review that follows will synthesize results from studies derived from each of these paradigms.

The final section of the research review discusses studies related to procedural aspects of the present investigation. This section includes a review of studies pertinent to the choices made concerning the selection of interview tasks and the conditions of administration of the tasks, e.g., the availability of manipulatives. It also presents background literature related to the individual interview procedure.

Classification of Addition and Subtraction Problems

Abstract addition and subtraction problems can be unambiguously classified into distinct problem types according to the form of the equation or open sentence. Specification of the operation and the position of the placeholder completely determines the problem type for simple abstract addition and subtraction problems. Six abstract problem types (and their symmetric forms) appear in the studies reviewed: a+b=__, a+__=c, __+b=c, a-b=__, a-__=c, and __-b=c. These problems can be classified either by the placeholder position and operation symbol in the problem or by the operation required for solution of the problem. Table 1 presents these two classifications. Rather than using a surface structure such as the operation symbol given in the problem, Moser's (Note 2) criterion for classifying problems as addition or subtraction problems was adopted. It consists of determining the operation which, when applied to the two numbers given in the problem, produces the correct answer. According to this criterion only the first and last (a+b=__ and __-b=c) of the preceding six abstract problems are addition problems; the other four require subtraction.

Table 1

Classifications of Abstract Problems

		Operation Symbol in the Problem	
		+	-
	Third-position	a+b=__ canonical addition	a-b=__ canonical subtraction
Placeholder Position	Second-position	a+__=c second-position missing addend	a-__=c missing subtrahend
	First-position	__+b=c first-position missing addend	__-b=c missing minuend

---

Operation Required for Solution of the Problem *

Addition	Subtraction
a+b=__ canonical addition __-b=c missing minuend	a-b=__ canonical subtraction a+__=c second-position missing addend a-__=c missing subtrahend __+b=c first-position missing addend

*classification used in the present study

Classification of verbal addition and subtraction problems is more complex. Van Engen (1949, 1955), Reckzeh (1956), and Gibb (1956) discussed "additive" and "subtractive" situations and distinguished "take away," "comparison," and "joining (additive)" as three types of subtraction. Other authors, e.g., Steffe (1970), have classified addition situations into those involving a transformation of the given sets (action) and those without a transformation (static).

Recently, comprehensive attempts have been made to specify distinctions between various types of verbal addition and subtraction problems. Five such attempts can be identified: a) The semantic analysis of Greeno and his colleagues (Greeno, 1980; Heller, Note 3; Riley & Greeno, Note 4); b) Moser's (Note 2) characterization of verbal problems in terms of entities, relationship or action, and characteristics of the question; c) Vergnaud's analysis of verbal problems which is based on the concepts of measure, time transformation, and static relationship (Vergnaud & Durand, 1976; Vergnaud, 1981); d) Kossov's classification of verbal problems into types of direct and indirect problems; and e) the linguistic analysis of verbal problems done by Nesher and her colleagues (Nesher, 1981; Nesher & Katriel, 1977, 1978; Nesher & Teubal, 1975).

These five categorizations of verbal problems essentially agree on major classification variables but differ in the attention they pay to details and subcategories related to each of these variables. Greeno's (1980) Change, Combine, and Compare schemata are more general semantic structures that are subsumed by Moser's ~~(Note 2)~~ Joining and Separating, Part-Part-Whole, and Comparison/Larger and Comparison/Smaller categories. Vergnaud's (1981) analysis of verbal problems is similar to Moser's but it admits additional problem types by including transformations and static relationships (both positive and negative) as separate categories of entities. Examples are "Bill wins 3 marbles. He plays another game and loses 1 marble. Altogether how many marbles did he win or lose?" and "I owe Mary 7 marbles and she owes me 4 of them. Altogether how many are owed to whom?". Kossov's (1975) distinction of problems by the larger/smaller dimension and by the type of corresponding open sentence (canonical or non-canonical) relates directly to Moser's. Nesher and Katriel (1978) further distinguished between action and static problems according to linguistic considerations such as sequential order of the text strings in comparison to the temporal order of the events described by those strings. The analyses developed by these authors are similar enough so that their related empirical work can be interpreted within the framework of a single classification of problems.

Change, Combine, Compare, and Equalizing are four classes of simple addition and subtraction problems that result from the preceding classifications. Change/Join and Change/Separate problems involve a change (increase and decrease, respectively) in an initial quantity over time. For each of these, three types of problems result depending on which quantity is unknown; for example, Join, Join/Change Unknown, and Join/Start Unknown are three distinct problem types.

Combine and Compare problems involve static relationships between a set and its two, disjoint subsets and between two distinct, disjoint sets, respectively. Combine problems require finding the union and Combine/Part Unknown problems involve finding one of the subsets. There are six types of Compare problems depending on whether the referent set, compared set or difference are unknown and whether the referent set is larger or smaller than the compared set.

In Equalize problems it is necessary to make one set equal to another. Thus, these problems involve comparison of two disjoint sets as well as the types of action found in Change problems. Three Equalize/Join and three Equalize/Separate problems can be generated by varying the unknown quantity. These problems are less common than Change, Combine, and Compare problems. Examples of these and other verbal problems are given in Table 2.

Classification of Strategies

A number of studies have discussed the procedures children use to solve verbal addition and subtraction problems (for example, see Carpenter & Moser, 1981; Gibb, 1953; Hebbeler, 1977) as well as abstract problems (for example, see Beattie, 1979; Groen & Poll, 1973; McLaughlin, 1935; Riess, 1943). These and other authors have used numerous terms to describe children's strategies--recalling a known addition or subtraction fact, counting, using derived facts, partial counting, guessing, and counting-on. The ambiguity in many of the above terms dictates that a standard terminology be chosen to identify the strategies children use to solve addition and subtraction problems.

Several researchers have described children's strategies for solving addition and subtraction problems in terms of an ordinal scale (Gibb, 1953; Moser, Note 5; Hatano, Note 6; Shchedrovitskii & Yakobson, 1975). The three levels, concrete representation, counting, and mental strategies, described by Shchedrovitskii &

Table 2

Selected Verbal Problem Types

Problem Type	Example
Change	
Join	Kim has 3 cards. His father gives him 6 more cards. How many cards does Kim have altogether?
Join/Change Unknown	Kim has 3 cards. Hom many more cards does he have to put with them so he has 9 cards altogether?
Separate	Lee had 7 toys. She gave 3 toys to Fran. How many toys did Lee have left?
Combine	
Combine	Leslie has 2 sugar donuts and 6 plain donuts. How many donuts does she have altogether?
Combine/Part Unknown	Chris saw 6 animals. Four were tigers and the rest were elephants. How many elephants did she see?
Compare	Joe has 3 fish. Mike has 9 more fish than Joe. How may fish does Mike have?
(There are five other types of Compare problems.)	
Equalize	There are 3 boys and 8 girls on the basketball team. How many more boys have to be put on the team so there will be the same number of boys and girls?
(There are five other types of Equalize problems.)	

Yakobson, Moser, and Hatano are useful for delineating classes of strategies that represent qualitatively different approaches to solving problems. Concrete representation strategies are the least abstract and rely heavily on the use of physical objects. Counting strategies are somewhat more abstract in that they involve the use of a sequence of counting words rather than the objects themselves, and mental strategies are the most abstract since they rely entirely on recalling addition or subtraction facts or mental manipulation of such number facts. Inappropriate strategies such as guessing or making no attempt to solve the problem comprise the fourth and lowest level of strategies.

In order to give examples of specific strategies that are typical of those in the preceding levels it is necessary to adopt standard terms to describe these strategies. Carpenter and Moser (1981) present a detailed analysis of the strategies children use to solve addition and subtraction problems. Theirs is the most comprehensive such attempt, and since it subsumes nearly all strategies identified by other researchers, their terminology will be used in the brief description of strategies which follows and in the more comprehensive definitions of strategy categories used in the present study that appear in Chapter III.

Counting All, Separate From, Adding On, and Matching are some examples of strategies that concretely represent the problem. In each of these the child uses physical objects such as manipulatives or fingers to construct sets which model the data and/or the relationship or action in the problem, and then counts or subitizes the resulting set to determine the answer. The Counting All strategy involves modeling the two sets or numbers given in the problem and counting the union set to find the answer. When solving subtraction problems children can use concrete representation in several ways. The Separate From strategy involves construction of one set of objects from which another set is removed, followed by counting of the remaining set of objects. The Adding On strategy is used when the child constructs a set of objects representing the subtrahend, increments that set of objects until its numerosity equals that of the minuend and then finds the answer by determining how many objects were added on. Matching is another strategy children use to solve subtraction problems (Carpenter and Moser, 1981). This strategy is based on the attempt to construct a one-to-one correspondence between two collections of objects representing the numbers or sets given in the problem. The two sets are matched one-to-one and the number of remaining unmatched objects determines the child's answer.

Children also use Trial and Error ~~to solve~~ addition and subtraction problems. This concrete representation strategy was not observed by Carpenter and Moser (1981) but has been documented in several other studies (Lindvall & Ibarra, Note 7; Rosenthal, 1975; Rosenthal & Resnick, 1974). This strategy most often occurs when the initial set is unknown, e.g., with __+b=c or __-b=c, or with a verbal problem such as "Billie had some toys. She gave 3 to Josie. Now she has 4 left. How many toys did Billie have to begin with?". Trial and Error is evidenced when the child models one of the sets in the problem with an arbitrary set of objects, performs the manipulations dictated by the remaining information in the problem, and then checks to see if the required final state exists. If not, the initial arbitrary amount is incremented or decremented as appropriate, and the process is repeated until a satisfactory final state is achieved. The most recent initial amount then represents the answer.

Children who do not physically model each of the numbers or sets in a problem often use forward or backward counting sequences to determine the solution. When using these, children begin the counting sequence at a number other than one, indicating a more abstract analog of the physical incrementing or decrementing of an initial set of objects. A child can begin the counting sequence with either the first addend given in the problem or with the larger of the two addends. In either case the forward counting sequence begins with one of the addends and continues for a number of counts equal to the other addend, ending at the sum of the two addends. Counting On From the Larger Addend represents a more efficient counting strategy since the minimum number of counts are needed to find the answer.

Counting strategies for subtraction problems can involve either forward or backward counting sequences. Two of the ways a child can solve a subtraction problem are by beginning the counting sequence from the minuend and counting backwards a number of counts equal to the subtrahend (Counting Down From) or by beginning the counting sequence from the subtrahend and counting forward until the minuend is reached (Counting Up From Given). In either case the child determines the answer by keeping track of the number of counting words uttered.

The most abstract strategies children demonstrate are those involving mental manipulations of numbers rather than physical manipulations of objects or the use of counting sequences. Children often remember basic addition or subtraction facts and either use these directly to solve the problem, or indirectly use a known fact to derive another needed fact. For example, when using such a strategy a child might generate the solution to a problem which requires the sum 5+6 by reasoning "5+5=10 so 5+6 must be one more, or 11."

Performance Prior to Formal Instruction

Preschool Children's Performance

The performance of preschoolers on addition and subtraction tasks provides a description of the capabilities of children at an age prior to that of the subjects used in the present study. This background is useful for viewing the development of quantitative skills, the ability to solve various types of addition and subtraction problems, and the evolution of children's strategies for solving these problems.

"Pre-numerical" tasks. One category of research at the preschool level involves "pre-numerical" addition and subtraction. This research views children's early notions of addition and subtraction as "non-quantitative" in that children focus on "more" or "less" rather than "how many." Pre-numerical or non-quantitative addition and subtraction tasks involve determination of whether or not two sets are equivalent, and if they are not, which set has "more." These tasks differ from addition and subtraction problems that require quantification, i.e., tasks in which one must determine how many objects are in the resulting set after addition or subtraction has been performed. Pre-numerical addition and subtraction tasks embody a unary conception of the operations of addition and subtraction (Weaver, 1981) in that they focus on the transformations which alter numerosity or the "physical manipulations creating inequalities" (Brush, 1978, p. 44). In Brush's study tasks were presented concretely using sets of objects and did not require the subject to specify the numerosity of sets, but rather whether or not two sets were of equal numerosity after one or both were transformed by appending or removing objects. Brush's study indicated that most preschool children understand that "adding to" and "taking away" alter the numerosity of a set of objects.

Gelman and Gallistel (1978) also presented evidence that preschoolers treat addition and subtraction (in the sense of "adding on" or "taking away") as number-relevant transformations. Starkey and Gelman (1981) described preschoolers' understanding of pre-numerical addition and subtraction as including four principles. Two of these are recognition that appending objects to or removing objects from a set increase or decrease numerosity, respectively. The other two principles related to addition and subtraction implicitly understood by preschoolers are inversion and compensation. These involve the use of appending objects to reverse the effect of removing objects (and vice versa) and reinstatement of a numerical relationship between two sets following a transformation of one of them by performing the identical transformation on the other.

Numerical tasks. Ginsburg and Russell (Note 1, Note 8), Hebbeler (1977), McLaughlin (1935), and Starkey and Gelman (1981) investigated preschoolers' performance on numerical addition tasks (those requiring precise quantification). McLaughlin found that few 3-year-olds could determine the total number of elements when two sets were combined but that many 4-year-olds were able to use counting to determine the total number of elements. The studies by Ginsburg and Russsell, Hebbeler, and Starkey and Gelman indicated that when the experimenter used objects to model each set described in the problem, preschoolers could correctly solve half to two-thirds of the problems, but that few children could solve addition problems when objects were not used or were screened after their initial presentation. The preceding studies identified guessing and counting (after problems had been concretely represented) as preschoolers dominant strategies, although number facts were recalled occasionally.

Groen and Resnick's (1977) study demonstrated that preschool children could be _taught_ to solve abstract addition problems with sums less than ten. Following extensive instruction and practice in using the Counting All strategy the criterion of perfect performance was reached. Latency analysis of subsequent responses suggested that several of the subjects spontaneously "invented" a counting-on strategy that was more efficient than the strategy they had been taught.

Summary. Previous research on addition and subtraction with preschool subjects indicates that young children understand operations on sets of objects which decrease numerosity and that if numerical addition and subtraction problems are presented concretely, i.e., the experimenter forms sets of objects representing one or both of the numbers in the problem, at least some preschool children can solve them. Little success has been reported on verbal problems presented in the absence of objects or on abstract problems (McLaughlin, 1935). The strategies used by preschoolers are often inappropriate ones, although some children display an ability to use and even invent counting strategies, especially in the presence of objects. One can conclude that preschoolers have a knowledge base which in some cases is adequate for addition and subtraction strategies to develop prior to kindergarten entrance.

Kindergartners' Performance

Abstract problems. Although a number of studies have investigated kindergartners' performance on certain addition and subtraction tasks, few have systematically documented their ability to solve abstract problems. Only one study (McLaughlin, 1935) reported the difficulty of addition problems for kindergartners. The 5-6 year-olds in her study correctly solved 38% of the abstract canonical (a+b=__) addition problems with sums less than 10. Ilg and Ames (1951) reported the only data on strategies used by kindergartners on abstract addition and subtraction problems. They noted that addition strategies progress during this age range from Counting All, to Counting On From the First Addend, to Counting On From the Larger Addend. Separating From and Counting Down From were the principal subtraction strategies, although Ilg and Ames reported some use of Derived Fact strategies involving addition facts (primarily doubles). Both of the preceding studies indicated that some kindergartners were able to solve addition and subtraction problems presented in the abstract context. Ilg and Ames' data indicated that counting strategies occurred both with and without objects present and that strategies similar to those observed in the studies with first-graders discussed subsequently also were present among kindergartners.

Verbal problems. Results from Grunau (1978), Ibarra and Lindvall (Note 9), Riley (Note 10), Schwartz (1969), and Shores and Underhill (1976) indicate that many kindergartners are capable of solving addition and subtraction problems, but that certain item types are considerably more difficult than others. Across the preceding studies Combine, Join, and Separate problems were less difficult than Combine/Part Unknown, Join/Change Unknown, and Compare problems. For example, p-values for Join, Separate, and Combine problems were often in the .45 to .80 range, whereas Join/Change Unknown, Compare, and Combine/Part Unknown problems often had difficulty indices in the .10 to .35 range. Another conclusion from the preceding studies is that problems without pictorial or manipulative aids generally had p-values .10 to .25 lower than those presented with aids.

Three studies (Ginsburg & Russell, Note 1; Hatano, Note 6; Hebbeler, 1977) used verbal addition and subtraction problems with kindergartners and, to some extent, gave attention to aspects of children's solution strategies. Hebbeler reported that kindergartners used appropriate strategies for the "overwhelming majority" of the addition problems presented and that counting and use of number facts accounted for approximately 70% and 10% of their strategies, respectively. Ginsburg and Russell's kindergarten subjects used appropriate strategies on over 70% of the Join items, with Counting All accounting for 55% of the strategies when no objects were present and other appropriate stategies such as use of number facts or Counting On From the Larger Addend accounting for another 17%. Hatano found that Japanese kindergarten children seldom exhibited any observable sign of counting but employed a type of Derived Fact strategy based on the use of 5 as an intermediate unit. This strategy involved mental regrouping in which numbers greater than five were regrouped as 5+x, where x<5. For example, 7+7 is 14 because there are two 5's and two 2's after regrouping. In each of these three studies at least some of the strategies used by older subjects were found in the kindergartner's repertoire as well.

One study focused on the procedures kindergartners use to represent or model verbal addition and subtraction problems. Lindvall and Ibarra (Note 7) identified difficulties young children have in modeling the numbers and relationships or actions in verbal addition and subtraction problems. They observed a trial-and-error approach to modeling problems involving a missing addend, missing minuend, or missing subtrahend, as well as an approach in which no attempt was made to model the unknown directly. Lindvall and Ibarra contended that kindergartners' main source of difficulty in modeling verbal problems was the identification of the set representing the answer to the problem. They found that difficulties in "tagging" the sets used to model the problem led to errors such as responding with one of the numbers given in the problem. The problems on which Lindvall and Ibarra identified difficulties in modeling were some of the same problems that were the most difficult in other studies.

Abstract and verbal problems. Only two studies have generated information pertinent to comparison of verbal and abstract problem performance at the kindergarten level. Each of these studies is severely limited. Woody (1931) reported 15-40% success rates for canonical abstract items with sums less than 10 and less than 15% on items with larger addends. He also reported administration of simple verbal problems, but no usable data are given for these items (Brownell, 1941). Consequently, no comparison, even of success rates is possible. Williams (1965) presented verbal problems with sums less than six to entering kindergartners. One item, 2+1=__, was presented in both verbal and abstract contexts, and performance was nearly identical (42% vs.

41% correct). Correct responses occurred on only 8% and 19% of the verbal problems modeled by 3+__=5 and 5-3=__, and the abstract item __+2=4 was answered correctly only 17% of the time. Neither of these studies compared strategies used on verbal and abstract problems.

Summary. Several general conclusions can be drawn from the preceding studies. Many kindergartners can understand and solve some simple verbal and abstract addition and subtraction problems prior to formal instruction on these topics. When problems are based on the canonical sentences such as a+b=__ and a-b=__, kindergartners can deal successfully with verbal and abstract items with sums less than ten roughly 25-50% of the time. No information is available as to the relative frequency with which children of this age can solve (or apply appropriate strategies to) problems in either the verbal or the abstract context but are not able to do so in the other context.

The rate of success is higher when objects are used and it is very high when the concretely presented verbal context (essentially enumeration) is used. Little attention was paid in previous research to systematic analysis of modeling procedures and strategies children use or to the effect of problem structure on choice of strategy. A majority of kindergartners appear to use appropriate strategies on verbal problems, but little is known about their strategies for abstract items. No systematic comparison of kindergartners' strategies and errors on verbal and abstract problems has been done.

Entering First-graders' Performance

Several studies which focused on first-grade subjects' ability to solve addition and subtraction problems have used entering first-graders, for whom it can be assumed that no formal instruction on addition and subtraction had taken place. The results of these studies can be compared more appropriately to those of the kindergarten studies reviewed previously than to studies using first-graders who had received initial instruction on addition and subtraction.

The results of several studies discussed subsequently and those from Carpenter and Moser's (1981) interviews with entering first-graders yield item difficulties comparable to those from studies which used kindergarten subjects. Buckingham and MacLatchy (1930) administered ten Join problems with sums less than 10 to entering first-graders. Twenty to 70% success was achieved with addends of 1 or 2. Hendrickson (1979) reported approximately 25% success on Join and Separate problems when the subjects were directed to model the first number in the problem prior to having the remaining portion of the problem read to them. Grant's (1938) tasks were similar to those of Buckingham and MacLatchy and also included Separate problems. His subjects' success rates ranged from 20-50%. Brownell (1941) administered a limited set of abstract and verbal addition and subtraction problems with sums and minuends less than 6 to entering first-graders. Difficulty indices ranged from .29 to .54 on abstract problems and .37 to .52 for the two verbal problems.

Carpenter and Moser (1981) longitudinally documented problem difficulty and strategy use across grade one on two verbal addition (Join and Combine) and four verbal subtraction problems (Separate, Compare, Join/Change Unknown and Combine/Part Unknown), each of which was unique in terms of problem structure. Entering first-graders correctly solved one-half to three-fourths of both addition problems when sums were less than 10 or when manipulatives were available for problems with sums 11 through 15. Performance dropped to approximately one-third correct when no manipulatives were available for problems with sums 11 through 15. These subjects correctly solved only one-third to one-half of the Separate, Compare, Combine/Part Unknown and Join/Change Unknown problems when minuends were less than 10 or when manipulatives were available for problems with minuends from 11 through 15. In only one instance, for Separate problems with minuends less than 10 with manipulatives available, was performance appreciably above the 50% level (64% correct). When no manipulatives were available for problems with minuends 11 through 15, the entering first-graders correctly solved only 14-25% of the subtraction problems.

The Carpenter and Moser study is unique in that it provides the only extant comprehensive data on the solution strategies used by first-graders on addition and subtraction verbal problems. Carpenter and Moser's model of children's solution processes for verbal addition and subtraction problems contends that problem structure is the principal determinant of young children's choices of strategies for solving verbal problems; variations in problem structure can be shown to account for the observed variations in children's choices of strategies across various item types. In the Carpenter and Moser study the two addition problems, embodying action and static situations, elicited similar strategies. They reported that the overall pattern of responses for both problems was almost identical both in terms of number correct and strategy.

Problem structure was strongly related to strategy choice on the four subtraction problems, with the strategy most frequently used being that which most directly modeled the action or relationship described in the problem. The subtractive strategies (Separate From or Counting Down From) and additive strategies (Adding On or Counting Up From Given) were used nearly universally on the Separate and Join/Change Unknown problems, respectively. Matching was found to be the most frequently used strategy on the Compare problem when objects were available, and both additive and subtractive strategies occurred on the Combine/Part Unknown problem, with the subtractive strategies being the predominant ones. Although the Combine problem was somewhat ambiguous, there were clear differences in solution strategies for these four problems with differing structure.

Carpenter and Moser concluded that children's difficulties in figuring out how to model the relationships in static problems may have accounted for their being less consistent in their choice of strategy on the Compare and Combine/Part Unknown problems. Carpenter and Moser's data indicate that young children have independent conceptions of subtraction and that they are not aware of the interchangeability of their strategies. Children's initial approaches to solving verbal subtraction problems are tied very strongly to the actions or relationships in the problems.

Summary

Several conclusions can be drawn from studies that examined preschoolers', kindergartners' and entering first-graders' performance on addition and subtraction problems. It appears that in spite of a lack of instruction on the operations of addition and subtraction, some young children spontaneously "invent" and use appropriate strategies to solve addition and subtraction problems. These strategies often closely mirror the structure of the problems and often involve concrete representation of the problem rather than counting or recalling number facts, although strategies in the latter two categories are employed by some children prior to any formal instruction on addition and subtraction. Children are more successful when manipulative objects are present, yet some children have difficulty constructing a physical model which represents the action, relationship or operation in the problem. The existence of low success rates on some problem types suggests that problems for kindergartners must be chosen carefully to ensure that useful information results from their administration.

Performance of First-graders After Instruction

Mid-year or end-of-year first-graders typically have experienced initial instruction on the operations of addition and subtraction. Thus, their responses to addition and subtraction problems can be expected to be different from those of younger subjects who have not had such instruction.

Abstract Problems

Beattie and Deichmann (1972), Groen and Poll (1973), Houlihan and Ginsburg (1981), Lindvall and Ibarra (1980), and Weaver (1971) provide data on the difficulty of various abstract addition and subtraction problems administered to mid- and end-of-year first-graders. Different problems and conditions of administration, e.g., the inclusion of problems with no solution in the Weaver study, may account for some differences between these studies. Nevertheless, across these studies, differences in difficulty among the six simple open addition and subtraction sentence types (a+b=__, a-b=__, a+__=c, a-__=c, __+b=c, and __-b=c) were comparable. These studies indicate that the simplest open sentences are the canonical addition (a+b=__) and canonical subtraction (a-b=__). These consistently were solved correctly more than 60% of the time with addition being the easier of the two. The two missing addend sentences (a+__=c and __+b=c) and the missing subtrahend sentence (a-__=c) were next in level of difficulty with p-values ranging from .46 to .87. Weaver and Lindvall and Ibarra both found that the second-position missing addend problem (a+__=c) to be slightly easier than the first-position missing addend problem (__+b=c). The missing minuend problem (__-b=c) was decidedly more difficult than the other open sentences, with difficulty indices being approximately .25 or less in all but Beattie and Deichmann's study (which drew data from workbooks whereon children presumably received help).

Houlihan & Ginsburg reported that first-graders were successful on only 27% of the addition items involving one single-digit and one two-digit addend in spite of the fact that over 60% of them used an appropriate strategy. When both addends were two-digit numbers, less than 5% of the subjects were correct and approximately one-third used an appropriate strategy. This suggested that problems with two-digit addends were potentially too difficult for most first-graders and certainly too difficult for kindergartners.

Of the studies which have investigated the strategies first-graders use when solving abstract addition and subtraction problems, some have used direct observation and others have used inferential techniques. Brownell (1941), Peck and Jencks (1976) and Houlihan and Ginsburg (1981) directly observed children's solutions of addition, addition and subtraction, and missing addend problems, respectively. Houlihan and Ginsburg reported that over three-fourths of the first-graders in their sample used an appropriate strategy on two abstract single-digit addition problems. Of these strategies, counting was the predominant strategy, with approximately equal numbers of subjects using Counting All and the counting-on strategies, i.e., Counting On From the Larger Addend and Counting On From the First (Smaller) Addend. Brownell individually administered abstract addition and subtraction problems with sums and minuends less than 10 to first-graders at mid-year and again at the end of the school year. Recalling a number fact and guessing were the most frequently used strategies, although Derived Fact and counting strategies were also exhibited. Peck and Jencks gave no detailed report of strategies employed, but noted that approximately 80% of the children who could correctly solve missing addend problems used an overt counting strategy. Several children used mental counting and about 15% recalled a number fact. Although counting was the predominant successful strategy, only 60% of their total sample employed such a strategy. Peck and Jencks' study demonstrated that first-grade children who had experienced initial instruction on addition and subtraction could successfully solve missing addend number sentences, primarily by counting.

Studies using response latencies to infer children's strategies on abstract addition problems (Suppes & Groen, 1967; Groen, 1968) found that reaction times were a function of the smaller of the two adends, suggesting that the best-fitting model of children's early strategies for addition problems is Counting On From the Larger Addend. Groen and Poll (1973) found that the only counting model that fit observed latencies for missing addend problems was one in which the number of counts was determined by the relative efficiencies of counting up from the addend to the sum and counting down from the sum a number of counts equal to the addend, e.g., for 3+__=8, counting up five units from 3 is less efficient than counting down three units from 8. However, this model fit observed latencies only for the second-position missing addend problems (a+__=c). Groen and Poll's study did not present conclusive evidence that first-graders base their choice of solution strategy on considerations of efficiency. However, response latency studies with young children do suggest that at the time when children have had little formal instruction on the addition and subtraction operations, their performance can be modeled by strategies that involve counting.

Error analysis was used as a means of inferring solution strategies in several investigations with first-graders. Beattie and Deichmann (1972) globally classified errors in first-graders' workbooks as basic fact errors, incorrect operation errors, and unclassifiable ones. Only 7% of first-graders' errors on canonical addition problems entailed use of the wrong operation, and the corresponding rate of such errors for canonical subtraction problems was 24%. The data on basic fact errors can be combined with frequencies of correct solutions to infer the frequency of use of an appropriate strategy. Such a procedure yields a high incidence of appropriate strategies, perhaps inflated by the help which students may have received when doing workbook pages. Weaver's (Note 11) data on incorrect responses similarly can generate estimates of the use of appropriate strategies; his data for canonical addition items quite closely parallel Houlihan and Ginsburg's (1981) interview data for similar items. Although many errors appear to be systematic when computational algorithms are involved (Ginsburg, 1977b), systematic errors may be more difficult to identify solely from responses when less complex tasks such as simple addition and subtraction are invovled. Inferential error analysis can be inaccurate in classifying counting errors and contributes little to identification of the solution strategies children use.

Verbal Problems

Many studies have found that first-graders perform well on verbal addition problems after receiving instruction on addition and subtraction. Depending on the size of the numbers used in the problems and the availability of manipulatives, difficulty indices for first-graders have generally been greater than .50 and often as high as .80 or .90 on verbal addition problems (Brownell, 1941; Carpenter & Moser, 1981; Hebbeler, 1977). Verbal subtraction problems have been consistently more difficult than addition problems, with difficulty indices often below .50, athough LeBlanc's (Note 12) first-graders were successful on approximately 65% of Combine/Part Unknown problems.

Most studies that compared the difficulty of action and static addition problems at the first-grade level found that performance was not markedly different on these two types of items (Carpenter & Moser, 1981; Carpenter, Hiebert & Moser, 1981; Shores & Underhill, 1976; Steffe & Johnson, 1971). Steffe (1970) reported p-values of .85 and .77 for Join and Combine problems with first-graders. These findings differ somewhat from those of studies comparing the difficulty of action and static addition problems at the kindergarten level. Three kindergarten studies (Grunau, 1978; Ibarra & Lindvall, Note 9; Shores & Underhill, 1976) reported slightly better performance on the static Combine problem than on the Join problem. One might conclude that prior to instruction, Combine problems are as easy or perhaps even easier than Join problems, but that after instruction, these problems are essentially of equal difficulty.

Differences in difficulty between verbal subtraction problems have appeared in many studies. One trend in a number of studies at the kindergarten level and with older subjects was a distinction between the action subtraction problems, with the Separate problem being less difficult than the Join/Change Unknown problem (Ibarra & Lindvall, Note 9; Rosenthal & Resnick, 1974; Schell & Burns, 1962). Two first-grade studies, (Carpenter & Moser, 1981; Steffe & Johnson, 1971), however, reported that the Join/Change Unknown problem was less difficult than the Separate problem. Another difficulty trend among verbal subtraction problems involves action problems being less difficult than static (Combine/Part Unknown or Compare) problems. This trend appeared consistently (Carpenter, Blume, Hiebert, Martin & Pimm, Note 13), and difficulty indices between .10 and .50 for the static problems suggested that in the present study, less useful strategy data might be obtained from the kindergarten subjects and even the first-graders if static rather than action subtraction problems were used.

The strategies used by mid- and end-of-year first-graders on verbal addition and subtraction problems are varied and involve more sophisticated procedures than those used by kindergartners and entering first-graders. Hebbeler (1977) reported that counting strategies and use of number facts accounted for approximately 50% and 40%, respectively, of her first-grade subjects' addition strategies. She noted that the incidence of counting strategies was lower than that for kindergartners, that the presence or absence of manipulatives had little effect on children's addition strategies and that, in contrast to the preschool and kindergarten levels, guessing was practically non-existent among first-graders on simple addition problems.

The strategies reported by Carpenter et al. (1981) were consistent with Carpenter and Moser's (1981) data concerning the influence of problem structure on the solution strategies used by mid-year first-graders. Carpenter & Moser's mid- and end-of-year interview data were similar to their data on entering first-graders, although by mid- to end-of-year there was increased use of counting and mental (Number Fact and Derived Fact) strategies.

Carpenter (Note 14) compared first-graders' strategies for solving addition and subtraction problems prior to and after initial instruction on addition and subtraction. After instruction had taken place, first-graders generally used subtractive strategies (e.g., Separating From or Counting Down From) for all four types of subtraction problems. This contrasted with their use of strategies which quite closely mirrored problem structure prior to instruction. This shift after instruction to strategies which presumably reflect a unified conception of subtraction was not evident in Carpenter and Moser (1981).

Abstract and Verbal Problems

Three studies (Brownell, 1941; Lindvall & Ibarra, 1980; Steffe, Spikes & Hirstein, Note 15) administered both verbal and abstract addition and subtraction problems to first-graders. Steffe et al. reported similar performance on abstract "mental arithmetic" problems (.81 for addition and .54 for subtraction) and verbal problems (.78 for Join, .71 for Separate, and .48 for Join/Change Unknown). Brownell's mid-year first-graders performed better on verbal addition and subtraction problems than on abstract problems (.85 vs. .74), but these results were reversed (.84 vs. .92, respectively) for end-of-year first-graders.

Lindvall and Ibarra (1980) compared incorrect procedures used by first-graders (and entering second-graders) on the four non-canonical addition and subtraction open sentences and corresponding verbal problems (the structure of these was not clearly specified). Their categories of incorrect procedures included: use of the wrong operation, responding with a number given in the problem, "computational error" (presumably miscounting or incorrectly recalling a number fact), no attempt to solve the problem, and unclassifiable errors. One interesting result of this study was the difference in error types occurring on abstract and verbal problems. Although the total number of errors across the four item types was identical for verbal and abstract problems, use of the wrong operation occurred on 11% of the verbal problems and on 17% of the abstract problems. No attempt was made on 8% of the abstract problems and on only 1% of the verbal problems. However, children gave one of the given numbers as their answer more frequently on verbal problems (13.5%) than on abstract problems (.5%).

These differences in incorrect procedures suggest that children were more willing to attempt (or believed that they had some understanding of) verbal than abstract problems, and that they less frequently misinterpreted verbal than abstract problems based on non-canonical open sentences. Lindvall and Ibarra's research indicates that further and more detailed research is necessary to describe the incorrect solution processes used by kindergartners and first-graders on verbal and abstract canonical addition and subtraction problems. Since children have been shown to exhibit different errors on certain verbal and abstract item types, it is reasonable also to expect differences in their correct solution procedures.

Summary

The preceding review of studies using first-graders indicates that nearly half of entering first-grade children often correctly solve some verbal and abstract addition and subtraction problems, and that a number of additional children use appropriate strategies on these problems. Evidence from direct observation and response latency data indicate that counting strategies are used often, although perhaps not as universally as is inferred from response latency data. Children at the first-grade level seem to employ a variety of strategies, with choices among them being based on the structure of the problem or on the efficiency of a given strategy relative to that of other appropriate strategies. It is not clear whether strategy use becomes more unitary after instruction takes place or whether children continue to use many rather than a single strategy for different types of problems.

Certain item types are difficult for first-graders, in particular, those based on the open sentence __-b=c. Many first-graders can solve missing addend problems in abstract form. For the verbal problems findings have been mixed concerning the relative difficulty of Separate and Join/Change Unknown problems. The consistent finding that subtraction problems involving action are less difficult than static ones suggested that action problems be used in this study, since static problems might be sufficiently difficult to generate a considerable amount of non-useful strategy data (guessing or making no attempt to solve the problem).

Action and static addition problems have elicited similar performance from first-graders, although some first-grade (and kindergarten) evidence indicates that Combine problems initially might be less difficult. Similarly, some evidence exists (Brownell, 1941) that verbal problems initially might be less difficult than corresponding abstract problems, but that this difference might disappear after instruction on addition and subtraction takes place. Different error patterns have been observed on some verbal and abstract problem types; this evidence (Lindvall & Ibarra, 1980) suggests that children more frequently may apply the wrong operation on abstract than verbal problems. No study has compared first-graders' correct solution procedures for abstract and verbal addition and subtraction problems, and no study has examined the modeling procedures used by first-graders.

Older Children's Performance

A great deal of research on addition and subtraction has used subjects beyond the first-grade level. This research is pertinent to the present study for several reasons. First, this research provides an additional cross-sectional view of the development of children's solution processes for addition and subtraction problems. This is a useful aid to viewing the development of such solution processes within the kindergarten and first-grade levels. Second, it traces the difficulty of various problem types beyond the first-grade level, giving an indication of the capabilities of older children for solving item types other than those included in the present study. Finally, the research with older children includes several additional comparisons of performance on abstract and verbal problems.

Abstract Problems

Although children in grades two and above generally find all types of simple addition and subtraction problems less difficult than first-graders do, certain problem types remain difficult even for older children. For example, Weaver (1971), Grouws (1972) and Hatano (1981) reported that third-graders correctly solved only one-third to one-half of the open sentences of the form __-b=c with minuends less than 19. Difficulty indices for the remaining five open sentence types have been comparable to each other for children at or above the third-grade level.

A number of studies report data indicating that many of the strategies used by first-graders continue to be used by a non-trivial percentage of older children. Several studies employing response latency techniques to infer strategy use concluded that hypothesized counting strategies often fit observed latency data. Jerman (1970), Svenson (1975), and Svenson and Broquist (1975) concluded that Counting On From the Larger Addend was the strategy that best fit observed performance on simple addition problems in grades three through seven. Rosenthal's (1975) results with 9-year-olds were more ambiguous, with only 11 of 22 subjects being fit by any of the hypothesized models, but his study is important because it identified some subjects who presumably used a trial and error approach to solve open sentences such as __+b=c and __-b=c.

Of particular interest to the present study are the data reported by Woods, Resnick and Groen (1975) and by Groen and Poll (1973). These studies supported the hypothesis that older children base their choice of strategies on considerations related to the efficiency of various counting procedures. Several counting strategies might be used for problems such as 12-9=__ and 9+12=__, or 12-3=__ and 3+__=12. Data from these studies support the view that children would use Counting Up From Given for the two problems in the first pair and Counting Down From for the two problems in the second pair since these strategies would ensure that the child counted a minimum number of units. Woods et al. concluded that second- and fourth-grade children used a counting strategy based on the minimum of the smaller constant given in the problem and the difference between the larger and smaller constants given in the problem. Some of Groen and Poll's subjects were children aged 7-9 who were given missing addend problems; their repsonse latencies were also best fit by a counting model in which the number of counts reflected the minimum of the given addend and the difference between the sum and the given addend. These studies suggest that efficiency of the counting process is the dominant criterion used by children in grades one through four in their choice of counting strategies for abstract subtraction problems. This conclusion is different from Carpenter and Moser's (1981) finding that problem structure dominates children's choices of strategies on verbal subtraction problems.

A number of researchers have directly observed older children's strategies for solving abstract addition and subtraction problems by using individual interviews. Smith (1921), Thornton (1978) and Beattie (1979) observed counting and Derived Fact strategies being used by children in grades two through seven. Houlihan and Ginsburg (1981) observed that Counting All was used only infrequently by second graders on addition problems, but that counting-on strategies constituted nearly half of the appropriate strategies used by these children; recalling number facts and Derived Fact strategies comprised the remaining half of the appropriate strategies.

Brownell (1928) identified a range of individual differences among 14 children aged 7 through 9 who solved single-digit addition problems. Half of the subjects recalled number facts as their predominant strategy, three used number facts and Derived Facts with approximately equal frequency, two children primarily used counting, and two frequently used any of these three strategies. The descriptive analysis of individual differences in strategy use reported by Brownell provides an important precedent for the descriptive analysis in the present study. In another study Brownell (1941) documented strategies used by second-graders on abstract addition and subtraction problems. Counting-on strategies were used more frequently by second-graders than by first-graders, although counting occurred infrequently in comparison to Guessing, Derived Fact and Number Fact strategies. More than 20% of the second-graders' strategies on subtraction problems were categorized as a type of Derived Fact strategy; included in this category were use of doubles or known facts to generate other facts, use of addition facts for subtraction problems, and the use of a fact in commuted form, e.g., using 3+1=4 for 1+3=__.

Svenson, Hedenborg and Lingman (1976) reported that children aged 10-12 used number facts and counting-on strategies with equal frequency (on 36% of the items), with counting in units greater than one (16%) and Derived Facts (12%) comprising the remaining strategies on addition problems with sums less than 14. Svenson et al. concluded that subjects used "highly individual methods for solving some of the problems" (p. 169).

Lankford (1974) found that counting was the most frequently used strategy among seventh-graders for whole number addition problems and that 25% of these subjects used fingers and another 16% used marks or motions for tracking when counting. Lankford emphasized that even at the seventh-grade level "pupils vary widely in the computational strategies which they employ in operations with whole numbers" (p. 29).

When larger addends were used (Flournoy, 1957; Grouws, 1974; Russell, 1977), consistent evidence appeared for individuals' use of a variety of solution methods. Russell's third-grade subjects used a written algorithm only 50% of the time for canonical addition problems with sums between 19 and 48. Grouws' subjects used number facts, counting strategies, trial-and-error procedures, guessing, derived facts, and the standard computational algorithms to solve the four types of non-canonical open addition and subtraction sentences. Flournoy reported that approximately one-fourth of the third-graders used several methods for solving addition problems with a two-digit and a one-digit addend.

Verbal Problems

Hebbeler's (1977) second-graders correctly solved approximately 95% of simple addition problems and recalling number facts was their predominant strategy (used on 60% of the items). Children's performance on action and static addition problems improves with age and nears ceiling level in grade two and beyond, but second-graders still have difficulty with verbal subtraction problems (Gibb, 1956; Riley, Note 10; Schell & Burns, 1962), particularly the static Compare and Combine problems. Zweng (Note 16) found that counting and Derived Fact strategies continued to be used through grades three and four on verbal subtraction problems. Carpenter and Moser (Note 17) reported longitudinal data from end-of-year second-graders which indicated that number facts were used on less than half of the verbal problems with sums from 11 through 15 and that counting strategies continued to be used on 20-50% of those problems.

Abstract and Verbal Problems

Hirstein (1979) and Brownell (1941) each administered verbal and abstract problems to second-graders, but neither compared strategy use on problems in the two types of contexts. Brownell's second-graders performed slightly better on abstract than verbal problems. In Hirstein's study performance on Join and Separate problems and their parallel abstract problems was comparable. Verbal Join/Change Unknown problems were considerably more difficult (40% vs. 70% correct) than the first- and second-position missing addend problems (__+b=c and a+__=c). Hirstein also noted that performance on abstract and verbal problems was highly associated except for the missing addend problems. Subjects who were successful on verbal problems were also successful on abstract problems, and the converse was true, except that a large number of children who were successful on the abstract missing addend problems failed the corresponding verbal problems. "Passing" and "failing" were determined by correct answers rather than use of appropriate strategies, however.

Summary

The above research with older children has three implications for the present study. The first is that performance on simple verbal and abstract addition problems approaches ceiling level in grades two and beyond. Subtraction problems remain difficult, however, with 50-75% success on the static problems in grades two and three.

The second general result in the preceding studies is that the strategies identified with kindergarten and first-grade subjects continue to be used in later years. Counting strategies, use of number facts, and Derived Fact strategies seem to be the predominant strategies used by children beyond grade one. Whether or not the counting strategies are chosen by considerations of efficiency in the counting process or according to problem structure is an open question, although there is some empirical support for the hypothesis that older children attend to the efficiency of their counting procedures.

The final implication of the above studies is that there may be differences in children's ability to solve verbal and abstract problems, in particular, the missing addend problems (Hirstein, 1979). Whether differences also exist in the strategies children use to solve problems in these two contexts is at present unknown.

Conclusions

Previous research has documented the difficulty of many verbal and abstract addition and subtraction problems. The research using preschool subjects indicates that even before children enter kindergarten they can successfully solve simple canonical addition and subtraction problems. The fact that many kindergartners use appropriate strategies on addition and subtraction verbal problems suggests that, if carefully chosen, verbal and abstract addition problems are appropriate for kindergarten subjects. Performance has been found to improve after initial instruction on addition and subtraction, yet certain simple (single-step) subtraction problems remain difficult even for third-graders. Data on children's solution processes are needed to contribute to potential explanations of these difficulties as well as earlier ones which are exhibited by kindergarten and first-grade children. Although the relative difficulty of certain verbal and abstract problems for first-graders has been investigated (Brownell, 1941; Hirstein, 1979), research which provides a comparison of modeling procedures, solution strategies and errors on verbal and abstract addition and subtraction problems is conspicuously absent from previous research.

The present study provides data pertinent to the hypothesis that many young children possess a substantial repertoire of alternative procedures for solving addition and subtraction problems. Studies providing data on the processes children use for solving either verbal or abstract problems have indicated that already at the kindergarten level children use a variety of strategies on verbal problems and that these strategies often remain in use even at much higher grade levels. However, previous research has provided no comprehensive description of kindergartners' strategies for solving verbal and abstract problems, nor has any comparison of solution processes for abstract and verbal problems been done with children who have had initial instruction on addition and subtraction.

Detailed data on the solution processes used by kindergarten and first-grade children can supplement the existing data on the development of children's strategies for solving addition and subtraction problems. There is some evidence that children's initial strategies for solving verbal problems are influenced by the structure of the problem. When counting strategies are used the relationship of choice of solution strategy to either problem structure or efficiency in counting has been hypothesized and remains an open question, with competing hypotheses having been generated from studies employing verbal problems and those employing abstract problems. Although strategies involving concrete representation of the problem and counting strategies persist even after initial instruction on the operations of addition and subtraction, older children do appear to use more abstract strategies. These older children also may develop a more unified conception of each of the operations of addition and subtraction, treating subtraction problems with differing problem structures as instances of problems involving a single operation of subtraction rather than as distinct problem types. Data from the present study can provide information about how the solution procedures of children who have and have not had initial instruction on the operations of addition and subtraction differ on both abstract and verbal problems.

Research Related to Procedural Aspects of the Study

The studies reviewed subsequently served as background for construction and administration of the interview tasks. This section also will review the literature that served as background for the individual interview procedure chosen for this study. The difficulty of various abstract and verbal item types was a major consideration in choosing problem types to be used in the present study; other structural aspects of the tasks were influenced by studies discussed subsequently. Complete details of the selection of the interview tasks are given in Chapter III.

Number Size

Previous research indicated that children can solve problems better when the constants are small numbers rather than larger numbers. This finding held for both verbal problems (Carpenter & Moser, 1981; Vergnaud, 1981; Zweng, Note 16) and abstract problems (Grouws, 1972; Houlihan & Ginsburg, 1981). A considerable body of research amassed prior to 1940 attempted to determine the relative difficulty of the addition and subtraction combinations with sums and minuends less than 19 (for example, see Clapp, 1924; Knight & Behrens, 1928; Murray, 1939). These studies have been reviewed elsewhere (Brownell, 1941; Carpenter et al., Note 13; Suppes, Jerman & Brian, 1968). One consistent finding was that the difficulty of addition and subtraction combinations increases as the numbers get larger. This finding, along with Brownell's (1941) interview tasks which involved "taught" and "untaught" facts, suggested that a distinction be made for each of the kindergarten and first-grade levels between problems involving smaller, more familiar numbers and ones using larger, less familiar numbers. Houlihan and Ginsburg (1981) reported that abstract addition problems with two-digit addends were difficult for first-graders; this suggested a restriction of the size of the numbers in the tasks used in the current study to the "basic facts," i.e., addition problems in which both addends are single-digit numbers along with the corresponding subtraction problems. Verbal problems using basic facts less than 16 were found to be appropriate for use with first-graders in Carpenter & Moser (1981), and many studies with kindergartners (e.g., Grunau, 1978; Schwartz, 1969) have used items with sums and minuends less than 10.

Studies that have analyzed response latencies (e.g., Groen & Poll, 1973; Jerman, 1970; Suppes & Groen, 1967) have consistently reported uniformly lower latencies for problems involving doubles, e.g., 2+2 or 5+5. In terms of both difficulty and solution strategy, doubles appeared to be unrepresentative of the set of problems that can be generated using numbers which are basic facts. This suggested that doubles be excluded from the numbers chosen for use in the present study.

Symmetric Forms of Abstract Problems

Several studies have provided data pertinent to structural aspects of the abstract problems used in the present study. Weaver (1973) and Lindvall and Ibarra (1980) found abstract addition and subtraction problems with the operation on the left, e.g., 6+__=9, to be consistently easier than symmetric forms such as 9=6+__. Lindvall and Ibarra also reported that first-graders experienced difficulty reading sentences with the operation on the right.

Two studies have investigated the effect of horizontal or vertical format on the difficulty of abstract problems presented in written mode. Engle and Lerch (1971) and Beattie and Deichmann (1972) reported little difference in difficulty between problems in horizontal and vertical format.

Structural Characteristics of Verbal Problems

The effect on problem difficulty of different positions of the question in verbal problems and the ordering of the sentences within the problems has been investigated by several researchers. Rosenthal and Resnick (1974) varied the order in which temporal information was given in verbal problems modeled by a+b=__, a-b=__, __+b=c, and __-b=c. All problems given to the third-graders involved action described either in chronological order

> If Paul started out with 5 boats and he bought 3 boats, how many boats did he end up with?

or in reverse chronological order,

> How many boats did Paul end up with if he bought 3 boats and he started out with 5 boats?

They found the reverse order to be more difficult when percent correct was the criterion but not when latency of response was the criterion. Bolduc (1970) found that the position of the question (before or after the data) was not a significant factor in difficulty of addition problems for first-graders. Nesher and Katriel (1978) also found no difference in difficulty for children in grades 2-6 between verbal problems in which the order of the sentences reflected the natural temporal order of the occurrences in the problem and those in which the sequential textual order of the sentences did not reflect the temporal order of the occurrences.

The preceding studies suggested that in the absence of consistent data concerning the effect of the order of the sentences of the text in verbal problems that the preferred wording for verbal problems in the present study would be that reflecting the natural order of occurrences in the problem, especially since this form is comparable to that used in other studies (Carpenter & Moser, 1981; Ginsburg & Russell, Note 1; Steffe & Johnson, 1971).

Availability of Manipulatives

Much empirical attention has been focused on the effects of manipulative or pictorial aids on children's performance on verbal addition and subtraction problems. Results of many studies (e.g., Carpenter & Moser, 1981; Gibb, 1956; Steffe & Johnson, 1971) support the contention that performance improves when children have manipulative objects or pictures available. Pictures and manipulative objects have been found to have comparable effects on children's performance (Gibb, 1956; Ibarra & Lindvall, Note 9). The <u>only</u> evidence suggesting that manipulatives hinder performance is provided by both first- and second-grade results from Steffe et al. (Note 15) and Hirstein (1979). Moser (Note 5) does suggest, however, that the presence of manipulatives influences first-graders' choices of strategies, with fewer counting and mental strategies and more concrete representation strategies occurring when manipulatives are present.

Empirical evidence regarding the effects of manipulative or pictorial aids on the difficulty of abstract addition and subtraction problems is limited. Houlihan and Ginsburg (1981) reported that second-graders did not use manipulatives on single-digit or on single-digit, double-digit addition problems. This suggested that the present study include interview conditions in which manipulatives were both available and not available to the subjects, providing that kindergartners' performance was not influenced too adversely by the absence of manipulatives.

The availability of manipulatives is also related to the issue of the extent to which an interviewer uses objects to present problems to the subjects. The conditions under which manipulative or pictorial aids were used in previous research have varied widely, from simply making aids available to the subject, to requiring the subject to use manipulatives, to presentation of the problem via the experimenter's manipulation of the objects. In studies employing the latter condition (Ibarra & Lindvall, Note 9; LeBlanc, Note 12; Steffe, 1970) the experimenter used manipulatives to form sets representing the two numbers in the problem, performed necessary transformations of the sets (joining or separating) for action problems, and the subject was then required to determine the answer. This condition of presentation essentially reduces the problem to one of enumeration. One can argue that although the subjects are required to determine the answer, they are not required to actually solve the problem and are certainly not required to model any of the data, relationships or actions in the problem. Studies entailing such concrete presentation of problems by the experimenter have reported high rates of success, even among kindergartners. However, for the purposes of eliciting modeling and observing solution strategies, concrete presentation of the tasks by the experimenter was deemed inappropriate for the present study.

Oral Presentation of Tasks and Subjects' Reading of Written Abstract Problems

The studies reviewed subsequently are pertinent to the mode in which tasks were presented to the subjects, i.e., written or oral. Houlihan and Ginsburg (1981) found no differences in difficulty or in the strategies used by first- and second-graders between abstract addition problems presented in written mode ("4+3" written on a card) and those in oral mode ("How much is four and three?").

Studies using verbal problems with young children have relied exclusively on oral presentation of the tasks, but these problems have been read to subjects in two ways. Carpenter and Moser (1981) read the entire problem to first-grade subjects without pausing after each of the phrases in the problem, whereas Lindvall and Ibarra (Note 7) reported that kindergartners were unable to comprehend problems when they were read in their entirety. Lindvall and Ibarra's procedure entailed reading problems sentence-by-sentence and recording the child's modeling procedure after each sentence of the story had been read. The serialization imposed by such reading of the verbal problem potentially could have altered the child's strategy by precluding any approach in which the child would first model the final state described in the problem. Thus, line-by-line reading was inappropriate for the present study.

When abstract problems are presented in written mode, subjects often have difficulty reading the problem correctly and, consequently, solving the problem which is actually posed to them. Behr, Erlwanger and Nichols (1976) and Denmark, Barco and Voran (1976) documented young children's misreading and misinterpretation of written abstract addition and subtraction problems, and concluded that even after instruction children viewed "="

as an operator, i.e., an indicator that the operation present (the "+" or "-" symbol) in the open sentence should be performed on the two given numbers. Lindvall and Ibarra (1980) presented strong evidence that the ability to correctly read an open sentence was a prerequisite for being able to correctly solve it. The preceding studies suggested that in the present study written abstract problems should either be read to the subjects or read by the subjects and corrected prior to solution of the problem, thus ensuring that subjects would, indeed, solve the problem presented and not some other problem.

Children's Interpretation and Generation of Written Symbolism for Addition and Subtraction

Children's ability to interpret and generate the symbolism for addition and subtraction has been the focus of several researchers' efforts. Studies reported by Payne (1967) and Hamrick (1979) attempted to determine whether instruction on written symbolism for addition and subtraction should be delayed or introduced early in the school mathematics curriculum. Payne reported better achievement with early symbolization and Hamrick found that delayed symbolization led to better understanding for students who initially did not possess the prerequisites for understanding the symbolism. The present study somewhat differently addressed the issue of children's readiness for the symbolism used for addition and subtraction. Rather than assessing children's ability to profit from instruction related to symbolic representation of addition and subtraction, the present study assessed children's ability to solve and their processes for solving orally-presented abstract problems at the kindergarten level and written abstract problems at the first-grade level. Consequently, the studies by Hamrick and Payne, as well as those by Kennedy (1977) and Allardice (1977) on children's production of informal written symbolism for addition and subtraction, are peripherally related to the present study.

Written symbolism in the form of a number sentence is one type of model which a child might construct to represent the information in a verbal problem. Lindvall and Ibarra (1979) concluded that being able to solve a verbal problem was a prerequisite for being able to write a number sentence to model that problem. Carpenter (Note 14) also reported that few first-grade children could coordinate their solutions of verbal addition and subtraction problems with the number sentence they were required to write to represent the problem. Nearly one-fourth of West's (1980) third-graders could not write an appropriate sentence for a Join/Change Unknown subtraction problem. The difficulties identified in these three studies and in Nichols' (1976) subjects' attempts to use number sentences to represent actions on objects suggested that writing number sentences was often not helpful to young children's solutions of verbal problems and that subjects in the present study be neither required nor encouraged to write number sentences in conjunction with verbal problems.

Use of Individual Interviews

An individual interview to assess a child's performance on addition and subtraction problems can take several forms. Opper (1977) describes Piaget's clinical method, one diagnostic tool for studying children's reasoning. In a true clinical interview hypotheses are generated about the processes children use to arrive at their solutions and the subject's responses serve as a basis for subsequent tasks and questions from the interviewer. Opper also describes a modification of Piaget's clinical method which she terms the "partially standardized clinical method" (p. 92). This approach combines a degree of standardization with the flexibility of the clinical method by using standard tasks but allowing the interviewer freedom to be flexible in subsequent probing related to the child's response. This was the approach used in the present study.

Alternatives to the partially standardized individual interview exist and have been used by researchers to study various aspects of children's thinking. Naturalistic observation, teaching experiments, and the case study method (Opper, 1977; Easley, 1977; Stake, 1978) are three of these. However, each of these methods has advantages and disadvantages. Individual interviews do not generate responses that are as spontaneous as those which derive from naturalistic observation nor do they provide the depth and breadth of data found in the case study approach. On the other hand, the individual interview procedure minimizes occurrences of irrelevant behavior and provides an opportunity to focus on specific thought processes while retaining sufficient generalizability to make comparisons between subjects and tasks possible.

Researchers who have used the individual interview procedure with young children have often reported difficulty in eliciting or interpreting the child's verbalizations. Menchinskaya (1969) used thinking aloud and introspection to study problem solving behaviors of first-graders but reported that "verbal description of their actions was difficult even for the stronger pupils" (p. 25). Shchedrovitskii and Yakobson (1975) also reported difficulty in identifying first-graders' solution processes and focused on problems in which children could externalize their method of solution (problems presented with objects). Attempts to determine why a child chose a particular strategy or used a given operation in the computational process often have been unsuccessful (e.g., Zweng, Note 16). Thus, two critical aspects of the individual interview procedure are the choice of follow-up questions and the use of tasks that elicit solutions based on observable or easily inferable behaviors.

Opper (1977) pointed out some of the procedural difficulties associated with the individual interview method. Among these were the possiblity that the child will not be at ease and perform naturally in the course of dialogue with the interviewer, the problem of the interviewer maintaining neutrality and avoiding attempts to elicit "correct" answers, the child's misunderstanding of language that is not adjusted to the child's level, insufficient time for the child to reflect on the problem and to develop his/her explanations, and the necessity for the interviewer's correct interpretation of the child's actions and responses on which subsequent questions are based. Previous researchers' use of the partially standardized individual interview as a means of gathering data on young children's solution processes for both verbal and abstract addition and subtraction problems (Carpenter & Moser, 1981; Houlihan & Ginsburg, 1981) indicated that this method was an appropriate method for use in a status study such as the present investigation, providing the preceding difficulties with the individual interview method were recognized. Attempts to avoid or minimize these difficulties are described in Chapter III.

Summary

The studies discussed previously represent one of the two major sources of input into the procedures for construction and administration of the interview tasks. They provided background information relevant to the numbers used in the problems, to the format in which problems were presented and to the environment in which the problems were presented. A second source of input into the procedures for construction and administration of the tasks were pilot studies with kindergartners and first-graders. The findings of these pilot studies and the resulting procedures chosen for use in the present study are the focus of the next chapter.

Chapter III

EMPIRICAL PROCEDURES

The purpose of this study was to provide a description of kindergarten and first-grade children's processes for solving certain addition and subtraction problems and to compare their performance on items presented in abstract and verbal problem contexts. This chapter presents the questions related to the preceding purpose and the empirical procedures chosen to attempt to answer those questions. The procedures used were selected to identify solution processes. Children were individually interviewed, their strategies and errors were observed, and appropriate anecdotal data were gathered.

Research Design and Questions Addressed by the Study

The study was a cross-sectional status study in which the variables of interest were problem presentation context, number size, and problem type. The two grade levels, two presentation contexts, two number size levels, and six problem types are shown in Figure 1. Kindergarten and first-grade subjects were both given problems in two contexts, verbal and abstract. Within each context six problems involved small numbers and six involved larger numbers. A detailed description of the six problem types is given in a subsequent section

Kindergarten

	Verbal						Abstract					
	Problem Types						Problem Types					
	V1	V2	V3	V4	V5	V6	A1	A2	A3	A4	A5	A6
Small numbers (sums less than 6)												
Larger numbers (sums 6 through 9)												

First Grade

	Verbal						Abstract					
	Problem Types						Problem Types					
	V1	V2	V3	V4	V5	V6	A1	A2	A3	A4	A5	A6
Small numbers (sums 6 through 9)												
Larger numbers (sums 11 through 15)												

V1 - Join
V2 - Combine
V3 - Separate (small difference)
V4 - Separate (large difference)
V5 - Join/Change Unknown (small difference)
V6 - Join/Change Unknown (large difference)

A1 - Canonical addition (a+b=__)
A2 - Canonical addition (a+b=__)
A3 - Canonical subtraction (small difference) (a-b=__)
A4 - Canonical subtraction (large difference) (a-b=__)
A5 - Second-position missing addend (small difference) (a+__=c)
A6 - Second-position missing addend (large difference) (a+__=c)

Figure 1. Organization of Variables of Interest

dealing with the tasks used in the interviews.

The questions of interest fell into three categories. The first category consisted of questions concerning the description of children's processes for solving addition and subtraction problems and differences in their performance on abstract and verbal problems. Questions in the second category focused on the difference in performance of kindergarten children, who had not received initial instruction on addition and subtraction, and first-grade children, who had received initial instruction. The third category dealt with the description of individual differences in the solution strategies children exhibit when solving addition and subtraction problems. The questions that follow are worded as substantive research questions rather than statistical null hypotheses to be tested.

<u>Questions Related to Performance on Abstract and Verbal Problems</u>

Prior to comparing performance on abstract and verbal problems it is useful to characterize children's performance on problems in each of these two contexts. Thus, two questions in the first category involved description of children's strategies for solving verbal and abstract problems.

Question 1. What strategies do children

in grades K and 1 use to solve addition and subtraction verbal problems?

Question 2. What strategies do children in grades K and 1 use to solve abstract addition and subtraction problems?

The issue of relative emphases on problems in verbal and abstract contexts in initial instruction on the operations of addition and subtraction generated questions involving comparison of children's performance on abstract and verbal problems. Prior to instruction on addition and subtraction children have experienced physical situations involving joining, separating, equalizing, and comparing sets of objects. These processes form the basis for the problem structure distinctions that are possible among various verbal problems. If kindergartners base their choice of strategy primarily on problem structure one might expect performance to be better on verbal than abstract problems because cues to familiar processes are provided by the physical situation in verbal problems. As a consequence of typical first-grade instructional emphasis on addition and subtraction in abstract number sentence format, one might expect little difference in performance on abstract and verbal problems among first-graders. Measures of children's performance would be the frequency with which they use appropriate strategies and the percentage of problems solved correctly.

> Question 3. Within each of grades K and 1 are there differences between children's ability to solve addition and subtraction problems presented in verbal problem context and their ability to solve corresponding problems presented in an abstract context?

Children's performance on abstract and verbal problems can also be compared with respect to the strategies used to solve problems in the two contexts. Carpenter and Moser (1981) found that problem structure was related to the choice of strategy used on various verbal subtraction problems. A wider variety of problem structures exist for verbal problems than for abstract problems, e.g., for the abstract problem a-b=__ one can construct several verbal problems involving either action or static situations. Because a wider variety of problem structures occur for verbal than abstract problems, it was hypothesized that different strategies may be used for corresponding verbal and abstract problems.

> Question 4. Within each of grades K and 1 are there differences between the strategies children use to solve verbal addition and subtraction problems and those used for

corresponding abstract problems?

A further question concerned differences in performance on verbal and abstract problems with small numbers and those with larger numbers. Results from other studies (e.g., Moser, Note 5) suggested that, in both verbal and abstract contexts, children would use different strategies for larger number problems than for those with small numbers.

> Question 5. Are kindergarten and first-grade children's strategies for solving verbal or abstract problems different for problems with small numbers than for problems with larger numbers?

Another question was related to the counting strategies children use to solve verbal and/or abstract problems. Two conflicting hypotheses have been generated to account for children's choices of counting strategies on subtraction problems. Studies utilizing response latency methodology (Woods, Resnick & Groen, 1975; Groen & Poll, 1973) have concluded that children who use a counting strategy to solve an abstract problem base their choice of counting strategy (counting forward or counting backward) on the relative efficiency of the two counting methods. Carpenter and Moser (1981) used direct observation of strategies and concluded that for verbal subtraction problems in which the subtrahend was greater than the difference; when counting strategies are used, they reflect the structure of the problem. When children use a counting strategy to solve problems such as 8-6=__ and 8-2=__ or corresponding verbal problems, they can distinguish among problems by the size of the difference between the numbers (i.e., count up "6; 7, 8" for 8-6=__ and count down "8; 7, 6" for 8-2=__), or they can choose a strategy that mirrors the structure of the problem (i.e., count down "8; 7, 6, 5, 4, 3, 2" for 8-6=__ and count down "8; 7, 6" for 8-2=__). Since kindergarten children were expected to use concrete representation rather than counting strategies to solve subtraction problems, the following question concerning the influence of efficiency or problem structure was not posed for the kindergarten level.

> Question 6. Do first-graders who use counting strategies to solve verbal and/or abstract subtraction problems use strategies which mirror problem structure or strategies which reflect attention to the efficiency of alternative counting procedures?

<u>Questions Focusing on Grade K--Grade 1 Differences</u>

A second category of questions was derived from the two grade levels in the study. First-grade children who have experienced formal instruction on the operations of addition and subtraction were expected to employ more abstract solution strategies. More abstract strategies are exhibited when, rather than using strategies that concretely represent the problem data, children count on or count back or use a mental strategy such as recall of a basic fact. Differences in the level of abstraction of kindergartners' and first-graders' strategies for solving abstract and verbal problems have not been addressed in previous studies. Such differences are of interest because they address children's development of increasingly abstract and efficient strategies and can potentially influence the problem context used for initial instruction.

> Question 7. Are there differences in the level of abstraction of kindergartners' and first-graders' strategies?

First-graders who have received formal instruction on the operations of addition and subtraction may have a more unified and abstract concept of these operations than kindergartners who have not had instruotion. One way of demonstrating such a unified concept of the operations is by flexibly or interchangeably using strategies that directly model the problem and those that do not. For example, a child might use Separate From and

Counting Up From Given on two separate problems; the first directly models the structure of the problem while the latter does not. The following question addressed potential differences in the flexibility with which kindergartners and first-graders use strategies that directly model or do not directly model problem structure.

Question 8. Are there differences in the flexibility with which kindergartners and first-graders choose among alternative strategies reflecting and not reflecting problem structure?

Other questions derived from the cross-sectional aspect of the study pertained to children's errors and misconceptions of addition and subtraction. Aside from errors of omission (guessing or not attempting the problem), errors exhibited in the solution of addition and subtraction problems can be of two types, procedural errors or errors of interpretation. Procedural errors include errors such as miscounting and forgetting problem data, while errors of interpretation include use of the wrong operation or inability to correctly model the problem. Carpenter and Moser's (1981) analysis of first-graders' errors on verbal problems documented use of the wrong operation but did not detail children's inability to model problems. It was expected that as a result of instruction or lack thereof, kindergartners' errors would be qualitatively different from those of first-graders.

> Question 9. Are kindergartners' errors of interpretation qualitatively different from those exhibited by first-graders?

The frequency of occurrence of different types of errors on abstract and verbal addition and subtraction problems is unknown, especially at the kindergarten level. Quantitative differences may also exist between the errors of kindergartners and those of first-graders.

> Question 10. Do various types of errors in solving addition and subtraction problems occur with differing frequencies for kindergartners and first-graders?

Questions Pertaining to Individual Differences

The extent to which individual kindergarten and first-grade children differ in their capability for solving abstract and verbal problems is unknown. It is possible that some children can easily solve problems in one context but not in the other. Also, individuals may differ in their ability to solve addition problems and subtraction problems. The solution strategies children use determine another important dimension of potential individual differences. If one can meaningfully cluster children according to the types of problems they can solve and the patterns of solution strategies they use, instruction can be tailored to these individual differences.

Question 11. What individual differences occur among kindergarten and first-grade children in their ability to solve and their strategies for solving verbal and abstract addition and subtraction problems, i.e., within each grade level can interpretable clusters of children be formed according to the types of problems they can solve and the types of strategies they employ?

The remainder of this chapter describes the empirical procedures selected to address the preceding questions.

Pilot Studies

Two pilot studies were conducted prior to choosing the research procedures for the present study. In the first, six kindergartners and twelve first-graders solved a variety of addition and subtraction problems with sums and minuends less than 16. The tasks included abstract and verbal problems based on the two canonical open sentences, a+b=__ and a-b=__, and the four non-canonical open sentences, __+b=c, __-b=c, a+__=c, and a-__=c. Abstract problems were presented in a variety of modes, including written number sentences and written-oral form

(3+2 accompanied by "How much is three plus two?").

This study yielded four conclusions:

1) Three of the abstract problem types, __-b=c, __+b=c, and a-__=c, and corresponding verbal problems were difficult for first-graders. With kindergartners these items provided little useful data; subjects nearly always guessed or made no attempt to to solve the problem. This suggested that these problems not be used in the study.

2) Some kindergartners relied entirely on manipulatives to solve the problems; others were unable to use the objects at all. It appeared that kindergartners should be encouraged to use manipulatives.

3) It was feasible to use problems from the Carpenter and Moser (1981) study with kindergartners. Kindergartners used appropriate strategies on approximately half of the problems with sums less than ten. Both kindergartners and first-graders exhibited many of the strategies identified in the Carpenter and Moser study, and kindergartners' verbalizations were sufficient to determine the strategies they used.

4) Overall performance on verbal and abstract problems was comparable both in terms of correct answers and use of appropriate strategies, but differences occurred in the strategies used on individual items.

Also, orally presented, abstract problems were less difficult than written ones. This suggested that abstract problems should also be read to the kindergarten subjects.

A second pilot study with sixteen kindergarten children provided additional information pertinent to children's strategies and to presentation of the tasks. This pilot study yielded the following information:

1) None of the sixteen children could correctly read the three abstract items, a+b=__, a-b=__, and a+__=c. This suggested that all abstract items be read to kindergarten subjects. The pilot study also suggested that use of the word "plus" be avoided in the main study; kindergartners more easily understood the language "__ and __ are how many?".

2) Less than half of the children used concrete representation with fingers or manipulatives. This suggested that modeling be encouraged among kindergarten subjects by making manipulatives available for all problems and by including "warm-up" tasks involving the use of fingers and manipulatives. The lower success rate of kindergartners as compared to the first-graders in Carpenter and Moser (1981) also suggested that more strategy information could be obtained from the condition in which the highest probability of success would occur,

i.e., the "manipulatives available" condition.

3) Appropriate strategies were used on 25% of the abstract items and 53% of the verbal problems. This suggested that kindergartners' capability for solving verbal problems might exceed their capability for solving abstract problems.

4) Several errors not discussed in previous studies were identified. Faulty modeling of subtraction (modeling both sets and removing one of them), "sequence" responses such as "4-3 is 2 because 4, 3, 2", and "two-digit" errors such as "2+3 is 23" were encountered. These suggested that detailed anecdotal accounts of errors, particularly errors in modeling, be recorded in the main study.

Problems Used in the Interviews

Two standard sets of addition and subtraction problems were administered to each subject; one consisted of verbal problems and the other of abstract problems. Complete description of these problems requires specification of three aspects: the structure of the problems, their wording, and the numbers used in the problems.

Problem Structure

The term "problem" is used in the sense of "textbook type" mathematical problems (Barnett, Sowder & Vos, Note

18). Mathematical problem solving research typically uses the term "problem" only when its connection to an individual is specified, i.e., a task is a problem only when it cannot be solved routinely by the individual to whom it is posed, and that individual accepts the task as a challenge and attempts to solve it. In the present study addition and subtraction problems were defined by their structure rather than in relation to an individual who was attempting to solve them.

A simple distinction can be made between addition and subtraction problems. Those problems in which applying the operation of addition to the two numbers given in the problem produces the correct answer are defined as addition problems. Similarly, subtraction problems are those in which applying the operation of subtraction to the two numbers yields the correct result. Thus, even though the problem 3+__=8 contains the symbol "+", it is defined as a subtraction problem. This definition is consistent with Moser (Note 2) and Reckzeh (1956) but differs from other researchers' definitions of addition and subtraction problems (Van Engen, 1955).

Addition and subtraction problems can be grouped into two large categories according to the context in which they are presented to children; this presentation context typically is either abstract or verbal. Problems presented in an abstract context are ones in which the data in the problem are presented without being related to any physical referents. Examples of abstract problems are written number sentences such as 8-2=__ and oral questions such as "Eight take away two is how many?" Verbal problems are defined as problems in which the data are embedded in a physical situation, i.e., there are actions on or relationships among the entities or physical referents to which the numbers in the problem are related. For example, sets of toys are the referents for the numbers two and eight in the verbal problem "Bill has two toys. How many toys does he have to put with them so he has eight toys altogether?". In this study the term "verbal problems" refers to problems that are often called "word problems" or "story problems."

Distinct problem structures are created by varying aspects of the relationships or actions on the entities in the problem and by varying the unknown number or question in the problem. All addition problems used in the study were ones based on a+b=__. The addition problems were all constructed so that the second addend was the larger of the two. This was done so that it would be possible to distinguish children who used a counting strategy in which they simply began counting with the first number given in the problem from those who used a more advanced strategy of counting on from the larger of the two addends.

The abstract addition problems presented orally to the kindergarten subjects were all of the form "__ and __ are how many?", and the abstract addition problems presented to the first-graders were all of the form a+b=__. Previous research and the pilot studies indicated that, because of their difficulty, other addition problems such as __-b=c or c=__-b would be inappropriate for kindergartners. Hence, only one type of abstract addition problem was used.

The verbal addition problems consisted of two problems based on a+b=__ but with differing problem structures. These were Join problems entailing action on the problem entities and Combine problems involving static relationships among the problem entities. Examples of these are given in the sample verbal problem tasks in Table 3. The Join and Combine problems were selected because they provided action and static addition problems on which different modeling procedures potentially could be used. Other verbal addition problems, e.g., those based on __-b=c or c=__-b, are difficult for first-graders (Carpenter et al., Note 13) and were not used in a related study (Carpenter & Moser, 1981). Thus, the verbal addition problems in the present

Table 3

Sample Verbal Problems

Problem Type	Sample Problem
Addition	
Join	Judy had 3 stamps. Her mother gave her 6 more stamps. How many stamps did Judy have altogether?
Combine	Fred saw 2 tigers. He also saw 5 elephants. How many animals did he see altogether?
Subtraction	
Separate (small difference)	Mike had 6 kites. He gave 4 kites to Kathy. How many kites did Mike have left?
Separate (large difference)	Joan had 9 apples. She gave 2 apples to Leroy. How many apples did Joan have left?
Join/Change Unknown (small difference)	Susan has 6 cookies. How many more cookies does she have to put with them so she has 8 cookies altogether?
Join/Change Unknown (large difference)	John has 2 cats. How many more cats does he have to put with them so he has 9 cats altogether?

study were restricted to Join and Combine problems, those providing the greatest likelihood of eliciting useful data.

The subtraction problems were ones based on either the canonical subtraction sentence a-b=__ or on the second-position missing addend sentence a+__=c. These problems were ones on which first-graders were expected to have experienced instruction (a-b=__) as well as items on which they were expected to have experienced no instruction (a+__=c). Other subtraction sentences such as a-__=c were deemed too difficult for kindergarten subjects and items such as __+b=c were eliminated because of the modeling difficulties associated with problems in which the initial set was unknown.

The abstract subtraction problems presented orally to the kindergartners were of the form "__ take away __ is how many?" and "__ and how many are __?". The abstract number sentences presented to the first-graders were of the forms a-b=__ and a+__=c. At both grade levels within each type of subtraction problem a further distinction was made between problems in which the difference between the minuend and subtrahend was smaller than the subtrahend and those in which the difference was larger than the subtrahend. Henceforth, the former are referred to as "small difference" problems and the latter as "large difference" problems.

The subtraction problems used in the verbal problem interviews were Separate and Join/Change Unknown problems. These problems both involve action but have different structures; the Separate and Join/Change Unknown problems correspond to Gibb's (1956) "take away" and "additive" subtraction problems. Within each of these verbal problem types a distinction was again made between problems in which the difference between the minuend and subtrahend was smaller than the subtrahend and those in which the difference was larger than the subtrahend. Thus, four types of verbal subtraction problems were used in the verbal problem interviews. These problems are referred to as Separate (small difference), Separate (large difference), Join/Change Unknown (small difference), and Join/Change Unknown (large difference).

The four types of verbal subtraction problems were selected for several reasons. Instruction on subtraction is commonly introduced via separating sets of objects; thus, the Separate problem was appropriate for use with kindergartners and first-graders. The Join/Change Unknown problem provides a contrast to the Separate problem beacuse it is worded additively and may elicit different strategies. Both of these problems have been found to be less difficult than static subtraction problems such as Compare and Combine/Part Unknown problems (Carpenter et al., Note 13). The Separate and Join/Change Unknown problems also were chosen because they can be mapped unambiguously onto number sentences of the form a-b=__ and a+__=c. Such a mapping is not possible for Compare and Combine/Part Unknown problems but is necessary for comparing children's performance on verbal problems and their abstract counterparts. Small and large difference problems were included to provide pairs of problems for which different counting strategies were the most efficient (Question 6).

Format and Wording of the Problems

In previous studies children's performance on horizontal and vertical number sentences has been comparable (Beattie & Deichmann, 1972; Engle & Lerch, 1971). Since the subjects' instruction had utilized horizontal number sentences, horizontal format was used exclusively in the present study.

The wording of the Join, Separate, and Join/Change Unknown problems was varied by using several different nouns in each problem type. Different names such as Susan or Leroy and different objects such as kites or pencils were used. Two different types of Combine problems were used in each subject's verbal problem interview. One involved a reference to objects for which the subordinate and supraordinate classes were similarly named, i.e., sugar donuts, plain donuts, and donuts, and the other involved objects for which the supraordinate class carried a different name from that of the subordinate classes, i.e., tigers, elephants, and animals. Previous research (Bolduc, 1970; Kellerhouse, 1975; Steffe, Note 19) was inconclusive regarding the effect of such varying of the names of the objects in addition problems. Informal observations from the pilot studies suggested that children responded similarly to these two types of problems, and thus no attempt was made to vary problems systematically along this dimension. Appendix A contains the complete list of stems used to generate the verbal problems.

Assignment of Numbers to Problems

In each problem two numbers were given as part of the problem data. These two numbers were elements of a number triple (x,y,z) defined by x+y=z, with x<y<z. Two sets of number triples were chosen for each grade level. The criteria for selection of these number triples included:

1) "Small" numbers and "larger" numbers were included at each grade level so that some problems involved numbers with which the subjects had had

substantial experience and others involved numbers that the subjects had used less frequently. Number triples with the sum, z, less than six were designated as the small numbers for kindergartners and triples with $5<z<10$ constituted both the larger numbers for the kindergarten subjects as well as the small numbers for the first-graders. The larger numbers for the first-graders involved number triples with $10<z<16$. Since only a few number triples with sums less than six met the preceding criteria, it was necessary to use each of the three small number triples at the kindergarten level twice within each set of six problems involving small numbers.

2) Number triples involving doubles, e.g., (3,3,6), were not used because they generate problems which are less difficult and not representative of addition and subtraction problems using other number triples (Groen & Parkman, 1972; Svenson, 1975).

3) No addends were zero, and addends of one were included only for the small numbers at the kindergarten level.

4) Consecutive addends were avoided so that the difference between z-x and z-y was as large as possible. This offered the maximum likelihood that subjects would use different counting strategies on the subtraction problems z-x=__ and z-y=__ (see Question 6).

The number triples used for small numbers at the kindergarten level were (1,3,4), (1,4,5), and (2,3,5). The six number triples which served both as the larger numbers for kindergartners and the smaller numbers for first-graders were (2,4,6), (2,5,7), (2,6,8), (3,5,8), (2,7,9), and (3,6,9). The number triples used for the larger numbers at the first-grade level were (3,8,11), (4,7,11), (4,8,12), (4,9,13), (5,9,14), and (6,9,15).

Latin squares were used to generate six orders for the assignment of number triples to problems. Since only three number triples were available for use in problems with small numbers at the kindergarten level a partial Latin square procedure was used to assign the three number triples to the six interview problems at that level. Care was taken not to allow the same number triple to occur twice on any given type of problem, e.g., the triple (1,4,5) did not occur on both verbal or both abstract problems based on a+b=__. The number triple orders used in the study appear in Appendix B.

Construction of Problem Sets for Subjects

Six task orders were used to minimize any order effect in the administration of the interview problems. A modified Latin square yielded six different task orders for the six problems involving triples of a given number size level. The Latin square was modified so that no problem of the form a+__=c appeared as the first interview task at either of the number size levels. This was done so that the first problem a child received would not be unfamiliar. The six task orders appear in Appendix C. Four different task orders were used for each subject, one for each six-problem half of the verbal problem interview and one for each half of the abstract problem interview.

The item stems in Appendix A, the six number triple orders for each number size level given in Appendix B, and the six task orders in Appendix C were used to construct a set of problems for each subject. Since each subject was given four sets of six problems (verbal problems with small and larger numbers and abstract problems with small and larger numbers) it was necessary to ensure that no subject received any task order or number triple order more than once. Within this constraint, the task orders and number triple orders were uniformly and randomly distributed twice within each group of three subjects. For example, task order #3 appeared twice among the first three subjects as did number triple order #5. A deck of problem cards was prepared for each subject to ensure that the appropriate tasks were administered in the assigned order. Appendix D gives the specific verbal and abstract problems used in the interviews with one kindergarten subject and one first-grade subject.

Sample Selection

The sample consisted of 50 kindergartners (21 male, 29 female) and 54 first-graders (29 male, 25 female) from two rural/small town midwestern schools. Thirty-two kindergartners and 34 first-graders were from School A and 18 kindergartners and 20 first-graders were from School B. These 104 children comprised the entire population at these grade levels who had returned parental consent forms. The range of ages for the kindergarten sample was 5 years 5 months to 6 years 8 months, with a mean of 5 years 11 months. The range for the first-grade sample was 6 years 5 months to 8 years 4 months with a mean of 7 years 0 months.

Subjects were selected from the kindergarten and first-grade levels because kindergartners generally do not receive formal instruction on addition and subtraction while first-graders typically do receive such instruction. Schools A and B were chosen because of their willingness to participate in the study and because they differed from the schools used in a related investigation (Carpenter & Moser, 1981). Subjects in the present study differed demographically from the middle/upper-middle class subjects of the Carpenter and

Moser study; School A had extensive Title I programs and School B was predominantly rural. An attempt was made to choose schools that used a mathematics program different from Developing Mathematical Processes (DMP) (Romberg, Harvey, Moser & Montgomery, 1974). School A used Mathematics in Our World (Eicholz, O'Daffer, & Fleenor, 1978) and School B used Mathematics Around Us (Gibb & Casteneda, 1975; Bolster et al., 1975). These programs provided a contrast to the focus on verbal problem solving and use of manipulatives in DMP. This contrast enabled the present study to provide data concerning the generalizability of a portion of the Carpenter and Moser study to another sample.

Instructional Background of the Subjects

Information on the subjects' instructional backgrounds was obtained from teacher interviews conducted after the student interviews were completed. The teachers described emphases they had placed on sections of the text, supplementary activities used, and the extent to which manipulatives were available and used in the classrooms. All teachers reported that they attempted to use the approaches suggested in the texts.

Kindergarten children in School A had been instructed on "readiness work" for addition and subtraction. This consisted of exercises in which pictures showed two sets of objects, e.g., three red cars and two blue cars, and the child was required to determine the number of objects in each set and the total number of objects. "Subtraction readiness" problems that required the child first to determine the total number of objects and then the number of objects in each subset also were included. These exercises embodied a static interpretation of addition and subtraction focusing on relationships between the whole and its parts rather than active joining and separating of sets of objects. Kindergartners in School A had often used manipulatives in their mathematics activities but the use of fingers for counting had never been explicitly mentioned.

Kindergartners in School B had less experience with manipulatives and had done no formal readiness activities for addition and subtraction. This class had, however, discussed "ways of making" each of the new numbers they studied, e.g., when learning the number 6 they identified sets of 1 and 5, sets of 2 and 4, etc., with objects or pictures. "Taking away" and "adding on" had not been discussed.

In contrast to the kindergarten classes' emphasis on static representations of addition and subtraction the first-grade instruction in both schools had focused primarily on the actions of joining and separating sets of objects. In School B sets of objects were used for the initial introduction of addition and subtraction but most classroom work involved pictorial representations or abstract number sentences. Children in School B were not encouraged to use their fingers for modeling. Manipulatives and the number line were used frequently in School A, and in one of the two classes, children were encouraged to use their fingers when necessary.

The first-grade teachers in both schools indicated they had not included explicit instruction on counting strategies such as Counting On From the Larger Addend. However, they indicated that discussion concerning counting on and counting back took place with individuals as the opportunity arose. Suggestions such as "Couldn't you just start at 4 and say '5, 6, 7'" were offered to children who questioned the possibility of beginning the counting sequence at a number other than one. Suggestions about counting on and counting back were not systematically introduced nor were they made to all students.

Kindergartners in both schools had studied numbers up to ten. First-graders in both schools had done drill activities on addition and subtraction facts with sums and minuends less than ten. Children in School A occasionally had encountered problems with sums from 11 through 15, but no attempt had been made to commit these facts to memory. In both schools first-graders had received no instruction on missing addend problems. One twenty-minute lesson on writing number sentences to model verbal problems occurred in one first-grade class in School A; this was the only formal experience any of the subjects received on verbal problems of the type used in this study.

Interview Procedures

The partially standardized clinical interview (Opper, 1977) was used in the present study. Each item of a pre-determined set of problems was presented to the subject and the subject's response to the task was observed and coded by the interviewer. When the child's solution process or answer was ambiguous, the interviewer asked probing questions in an effort to elicit a clearer description of how the child solved the problem. The emphasis in this type of interview was on understanding how the child solved addition and subtraction problems. This method was selected because it provided both uniformity among the tasks and flexibility in the questioning used to clarify subjects' responses.

Administration of the Interviews

In the interval from March 10 to March 25 each subject was individually interviewed on two occasions, once to solve twelve abstract problems and once to solve twelve verbal problems. These interviews were done on separate days, often with more than one day's time intervening. The order of administration of the verbal and abstract problem interviews was counterbalanced so that a randomly selected half of the subjects at each grade level received a given type of interview first. Each day's interview lasted 15 to 30 minutes. Subjects were individually interviewed by one of ~~three~~ trained and experienced interviewers: the experimenter, a mathematics education graduate student, and a research specialist.

Mode of Presentation of the Problems

When problems are presented in either verbal or abstract context the mode of presentation can vary. For example, the mode can be oral, written, physical or concrete, or pictorial. The extent of the child's and the interviewer's involvement in the presentation of the problem varies across these modes. Orally presented problems are read by the interviewer while written problems are usually read by the child, and in the physical and pictorial modes the interviewer uses objects or pictures to model for the subject some of the data or relationships in the problem. When the interviewer models a portion of the problem for the child, the child's solution process may be influenced by the experimenter's actions and may no longer be spontaneous. Thus, the present study used neither the physical nor the pictorial mode for presenting problems to the subjects.

The addition and subtraction verbal problems were presented orally to both kindergarten and first-grade subjects due to the subjects' limited proficiency in reading. The verbal problems were read to the child from cards and re-read upon request as often as necessary to ensure that the child had an adequate opportunity to remember the numbers and relationships in the problem. Other studies (Carpenter & Moser, 1981; Ginsburg & Russell, Note 1) successfully used oral presentation of verbal problems and the pilot studies also indicated that such a presentation mode was appropriate for subjects in grades K and 1.

Verbal problems were read in their entirety rather than being read sentence-by-sentence with pauses designed to have the child model the information in each sentence immediately after it was read. Reading the complete verbal problem was consistent with Carpenter and Moser (1981) and in contrast to the procedures used by Lindvall and Ibarra (Note 7). Lindvall and Ibarra reported that kindergartners made few meaningful responses when problems were read without an extended pause after each sentence. However, one must be cautious of this result. Some of their item types were problems based on __+b=c and __-b=c, problems consistently difficult even for children beyond the kindergarten level. Also, the pilot studies demonstrated that many kindergartners could make meaningful responses to the verbal problems used in the study. Major pauses that are built into the reading of a verbal problem can influence a child's strategy. For example, if one reads, "John had 8 pennies. (pause, waiting for the child to construct a set of eight objects) He gave 6 pennies to Mary. (pause, waiting for the child to separate 6 of the 8 objects) How many pennies did John have left?", one discourages the use of an Adding On strategy, i.e., with the pauses included the child would be unlikely to solve this problem by constructing a set of 6 objects and adjoining additional ones until a set of 8 were formed. When the entire problem is read without pauses, the Adding On strategy at least becomes plausible. Consequently, no extended pauses were built into the reading of the verbal problems.

Abstract problems were presented orally to the kindergarten subjects because kindergartners in the pilot studies were unable to read number sentences such as 2+3=__. Oral presentation of both abstract and verbal problems made the mode of presentation of these problems comparable, although it did not provide a measure of kindergartners' ability to interpret abstract problems in written symbolic form.

Since first-grade instruction on abstract addition and subtraction typically employs written number sentences and since other studies (Lindvall & Ibarra, 1980; Weaver, 1971) presented abstract problems in number sentence format to first-graders, the written number sentence mode of presentation was used with the first-grade subjects. Houlihan and Ginsburg's (1981) results suggested that first-graders exhibit similar performance on oral and written abstract addition problems; thus the difference between the presentation modes of the verbal and abstract problems (oral and written, respectively) was not expected to influence first-graders' performance. Furthermore, when first-grade subjects were shown an abstract problem, they were required to read it prior to solving it, and their reading was corrected by the interviewer when necessary. Although this was done to ensure that the child was solving the intended problem rather than some misinterpretation thereof, it also served to make the presentation modes of the verbal and abstract problems more comparable.

Materials Available to the Subjects

Previous studies indicated that children's performance on addition and subtraction problems is better when manipulatives are available to model the problem data and relationships than when no objects are available. Making manipulatives available to the kindergarten subjects and encouraging their use was expected to increase the likelihood of obtaining useful data on their strategies and to aid in the interpretation of children's strategies. Because the problems were not expected to be as difficult for first-graders and because there are potential differences in the strategies subjects might use in the presence or absence of manipulatives, half of the first-graders were randomly selected to have manipulatives available during the interview. The remaining first-graders were not given objects, but, as with all subjects, were allowed to use their fingers or objects in their field of vision to assist in modeling or representing problems.

Subjects were neither required nor allowed to use paper and pencil to model the problems in written symbolic form. While it is true that problems can often be solved by translation into mathematical symbolism, such modeling is not easy for young children (Allardice, 1977). Carpenter, Moser and Hiebert (Note 20) found that many children when required to write number sentences to model problems, often did so only after solving the problem. Thus, the written mode of representation was not used.

Twenty cubes, ten orange and ten blue, were available to the kindergartners and to half of the first-graders. These were placed on the table after the warm-up tasks and it was suggested that the child could use them to help answer the questions. Whenever possible, cubes used by the child were pushed back into the pile after a problem was completed. This was done to avoid mistaken use of the results of a previous problem in an attempt to solve subsequent ones.

An "Oscar the Grouch" doll was used as a prop for the interviews with kindergarten subjects. Children were told that Oscar needed help listening to and answering questions about some number stories. At times children were encouraged to explain what they did "so Oscar also would know what to do."

Interview Protocols

Interviews took place in a room near the child's classroom. During the interview the child and the interviewer sat at a table that contained the materials used in the interview. After some initial conversation to put the child at ease, three warm-up tasks were administered. One was a sorting task designed to encourage the child to manipulate objects, to encourage verbalization, and to provide a "success experience." The second was a counting task designed to determine whether the child could accurately enumerate several sets of objects. The third required the child to construct sets of objects with required numerosity. Complete protocols for these tasks are given in Appendix E.

Care was taken to avoid any reference to addition or subtraction in the warm-up tasks. Although two of them involved counting and the use of cubes or fingers to represent sets of objects, no reference was made which tied these tasks to subsequent addition and subtraction problems. It was hoped that these tasks would encourage the use of objects or fingers by subjects who otherwise might have been reluctant to use them.

Following the warm-up tasks, the child was given two sets of six problems involving different number sizes; problems with the small numbers were administered first. Subjects were given as much time as they needed to solve each problem. When the interviewer was unsure of what the child had done, some general questioning techniques were used. This interaction varied from child to child and was dependent upon the child's actions and/or statements. An initial follow-up question was generally of the form, "How did you get that answer?" or "How did you decide __ was the answer?". Questions such as "Did you count?" were not used because such a suggestion of counting might have encouraged the child to describe a counting process even if one had not been used. Instead, questions such as "What were you thinking about when you did that one?" or "Were you thinking of any numbers to yourself?" were used. If children volunteered that they had been counting, then "Did you count forward or backward?" or "What number did you start counting with?" were appropriate questions. The questioning was designed to elicit explanations; when it confused the child or the child was unable to explain, the interviewer proceeded to the next problem. Protocols for the addition and subtraction tasks are given in Appendix E.

A final task, similar to the first warm-up task, involved sorting geometric pieces and was administered prior to the child's return to class. This provided another success experience and created a situation in which the last task performed before returning to class (and perhaps the one best remembered) was <u>not</u> an addition or subtraction problem.

<u>Coding of Responses</u>

During and/or after a subject's attempt to solve a problem four categories of behavior were coded.

Correctness of the response, attempts to model or represent the problem, the solution strategy, and errors were recorded along with pertinent anecdotal information. Appendix F contains the coding sheet used for the interviews. The preceding four categories are not mutually exclusive; the relationships among them are described in the following sections.

<u>Correctness of Response</u>

Subjects' numerical responses were coded as either correct or incorrect. When a subject obtained a correct answer by some incorrect procedure, e.g., obvious "wild guessing" or miscounting in which the second error offset the first, that response was coded as incorrect. When subjects stated that they could not do the problem and gave no numerical answer, No Attempt was recorded and the interviewer proceeded to the next problem.

<u>Model or Representation of the Problem</u>

A child models or represents a problem by changing the modality in which the components of the problem are given to some modality in which it is convenient to carry out the solution. Addition and subtraction problem components, i.e., the numerical quantities, the relationship and/or operation connecting the numerical quantities, and the relationship of the unknown quantities to the known quantities (Moser, Note 2), can be represented in modalities such as physical objects or pictures, written prose, abstract symbols, or internal mental imagery. In the present study problems were presented in verbal-spoken or abstract symbolic mode. Pilot studies and previous research (Carpenter & Moser, 1981) indicated that kindergarten and first-grade children often model or represent such problems either by using physical objects or by using no visible representation, in which case some form of mental representation presumably takes place. The coding categories for describing the models subjects employed are described in the following sections.

Cubes. Manipulative objects can be used in two ways. Cubes can represent sets described in the problem or numbers given in the problem. A set of four cubes might represent the "4" in the problem 4+__=11, or represent a set of four stamps in a problem that begins, "Susan had four stamps." The child's interpretation or representation of the action on or relationship between sets given in the problem or of the operation on the numbers given in the problem is realized by actions performed on the cubes.

Cubes can also be used in conjunction with a counting strategy that requires the child to keep track of the number of counting words uttered. For example, if a child begins counting from four and counts "five, six, seven, eight, nine, ten, eleven," seven cubes might be set out one-by-one to keep track of how many counts were uttered. In this case the cubes are not initially used to represent seven objects, but as counters to represent number words.

Fingers. Fingers also can be used in two ways in the modeling process. They can represent the sets of numbers described in the problem or they can keep track of the numbers uttered in a counting sequence.

No visible model. This category was used when no discernible model or visible representation of the problems was apparent from the child's actions or statements. In such instances it was inferred from subjects' behavior or explanations that they were either recalling memorized information such as addition or subtraction facts, using some mental manipulation of number facts or number properties, rhythmically keeping track of some counting sequence, or using some type of mental imagery.

Other models. Occasionally children visually or tactilely use other objects as models when representing a problem. This especially occurs when more than ten fingers are needed to represent the sets or numbers in a problem. Examples of other models are light bulbs, books on a shelf, floor tiles, or buttons or patterns on articles of clothing. This category also was used when children reported that they visualized a number line on the table top and used that to model the problem.

It is possible for a child to use more than one model to represent a single problem. In such instances two or more categories of model are coded. Children occasionally begin to use manipulatives or fingers and then abandon them in favor of another mode of representation. A decision was made to include the initial model only if it was apparent that the initial attempt was used in some way in the solution strategy ultimately employed. For example, if a child's actions when solving 4+__=11 entailed first forming a set of two or three cubes and then saying, "Four plus seven is eleven, so the answer is seven," the initial attempt to model the problem using cubes was judged not to have entered into the solution strategy and the category for "no visible model" was coded.

Solution Strategy

Once a child uses either a physical or mental mode of representation to model the components of an addition or subtraction problem, some action is performed on that representation. This action can be either physical or mental and constitutes the child's solution strategy.

Included in such mental actions are recalling an addition or subtraction fact or engaging in a particular type of counting. The physical actions that comprise solution strategies often involve manipulations of physical objects. These physical and mental actions on representations serve to characterize the processes children use to solve addition and subtraction problems. They are not intended to describe in detail *all* of the mental and physical processes a child uses when solving problems. For example, Case (1978) uses the term strategy to include processes such as looking at a numeral in a written problem, storing that symbol in memory, and so forth. Such a detailed level of description of strategies was not appropriate for the present study since it would not have enabled one to observe or confidently infer that a child was actually performing such actions. On the other hand, global strategy categories such as "adds given numbers" offer little useful information when describing differences in the processes children use to solve problems. The strategy categories in the present study represented qualitatively different solution procedures that could be observed or easily inferred from children's actions and explanations.

It was assumed that a subject used only one strategy to solve a problem. In some cases a strategy was initiated and then abandoned; in such instances the interviewer coded the strategy that yielded the solution of the problem. If no solution resulted, the last strategy used was coded. Coding categories for strategies were derived from pilot work and previous research (Carpenter & Moser, 1981). Strategies appropriate for addition problems are given first, followed by strategies for subtraction problems and strategies appropriate for both addition and subtraction problems. Strategies for addition problems are described in reference to the problem a+b=__ with a<b.

Counting All. Counting All is a strategy that concretely represents the problem. It involves construction of two sets, one for each addend. These are counted out "1, 2, ..., a" and "1, 2, ..., b." The union of the two sets is counted "1, 2, ..., a+b," and can be formed in three ways:

1) The union is formed incrementally as the child simultaneously models the second set and adjoins it to the first. The first set is formed and counted, the second set is modeled and counted while it is adjoined one object at a time to the first set, and the union is then counted. This is Counting All with one set.

2) The union is formed by adjoining the second set

en masse to the first. In this case the second set is counted before it is adjoined to the first set. This is Counting All with *two* sets.

3) The union is formed implicitly, but no joining action takes place. This is also Counting All with *two* sets.

Subitizing. Having modeled the two sets corresponding to the addends it is sometimes possible for the child to perceive the numerosity of the union set without having to count each member of that set. Subitizing occurs when the size of the union set is small, usually less than six or seven, or when fingers are used as the modeling device and the five fingers of one hand are perceived immediately.

Counting On From First (Smaller) Addend. In contrast to Counting All and Subitizing, in which sets or numbers in the problem are represented concretely, Counting On From First (Smaller) Addend employs a sequence of counting words to determine the solution to the problem. The counting sequence is forward, begins with the smaller addend or its successor, and ends with the sum, e.g., "(a), a+1, ..., a+b." The child knows when to stop by keeping track of the number of counting words recited; this tracking may be done mentally or by using fingers or objects.

Counting On From Larger Addend. This counting strategy is identical to Counting On From First (Smaller) Addend except the counting sequence begins with the larger addend or its successor, "(b), b+1, ..., a+b." This corresponds to the Min(a,b) strategy described in studies employing response latencies (Suppes & Groen, 1967). Counting On From First (Smaller) Addend and Counting On From Larger Addend were referred to as "partial counting" by Brownell (1941) and "counting on" by Steffe et al. (Note 15).

The strategies specific to subtraction problems are given next. These are described in reference to the problem a-b=__.

Separate From. The child uses manipulatives or fingers to construct the larger given set and then takes away or separates, one at a time, a number of objects or fingers equal to the smaller given number. Counting (or subitizing) the remaining objects or fingers yields the answer. Three counting sequences are used: "1, 2, ..., a", "1, 2, ..., b", and "1, 2, ..., a-b."

Counting Down From. This strategy is the counting counterpart to Separate From and involves the use of a backwards counting sequence beginning with or from the larger number (minuend) and involving as many counting number words as the given smaller number (subtrahend).

The counting sequence is either "a, a-1, ..., a-b+1" or "a-1, a-2, ..., a-b." In this strategy the child does not concretely represent the problem but uses a sequence of counting words to determine the answer.

Separate To. This strategy is similar to the Separate From strategy except that the separating continues until the smaller given quantity is attained rather than until it has been removed. Counting or subitizing the number of objects or fingers removed gives the answer. For example, a set of a objects would be formed, and a-b objects would be removed until b objects remained. Usually only two counting sequences are employed: "1, 2, ..., a", objects are removed (those remaining might be counted to check that b objects remain) and those removed are counted, "1, 2, ..., a-b."

Counting Down To. This strategy is the counting counterpart to Separating To and is similar to Counting Down From except that the counting sequence ends with the smaller given number. Two counting sequences are possible, "a, a-1, ..., b+1" or "a-1, a-2, ..., b."

Adding On. Adding On involves modeling the smaller given number (subtrahend), incrementing that initial set until the number of objects is equal to the larger given number (minuend), and then counting or subitizing the number of objects added on. Three counting sequences are used: "1, 2, ..., b", objects are added on while the counting continues "b+1, b+2, ..., a", and finally, the objects that have been added on are counted, "1, 2, ..., a-b."

Counting Up From Given. This strategy involves forward counting beginning from the smaller given number (subtrahend) and ending with the larger number (minuend). The child determines the answer by keeping track of the number of counting words uttered; this is done mentally or with fingers or objects. The typical counting sequence is "b+1, b+2, ..., a."

Matching. This strategy requires the use of concrete representation. The child forms two sets of objects, each set modeling one of the sets or numbers given in the problem. These sets then are placed physically or visually in one-to-one correspondence, and the answer is determined by counting or subitizing the unmatched objects. The counting sequences are: "1, 2, ..., a", "1, 2, ..., b", and "b+1, b+2, ..., a-b." Carpenter and Moser (1981) observed this strategy primarily on Compare problems. Since these problems were not included in the present study, this strategy was expected to occur infrequently, if at all.

Two mental strategies can be used for both addition and subtraction problems. These are described next.

Number Fact. This strategy is coded when the child produces the answer by recalling an addition or subtraction fact such as "nine minus three is six." This category is used when the child states the number fact or responds quickly with the answer and gives the justification, "I just know that." Number Fact is coded when either an addition or subtraction fact is used. If a child generates an incorrect answer by using an incorrectly recalled fact such as "two plus six is seven," Number Fact is coded, the incorrect fact is recorded, and "incorrect" is coded for correctness of response.

Derived Fact. This strategy involves mental manipulation of a known number fact to derive a number combination needed for determining the answer. Typical examples of the Derived Fact strategy are those using facts involving doubles, for example, "Six plus six is twelve, so six plus eight must be fourteen," and those using a combination involving ten, e.g., "I know that four plus six is ten, so four plus seven must be eleven." When this strategy is coded the interviewer records the specific explanation given by the child. Strategies of this sort have been labeled "roundabout procedure" by Smith (1921), "solving" by Brownell (1941), and "indirect memory" by Houlihan & Ginsburg (1981).

Several inappropriate strategies were coded for both addition and subtraction problems. These categories are described next.

Guess. This strategy is inappropriate and generally does not produce a correct answer. When a correct answer is achieved as a result of an obvious "wild guess," Guess is coded and correctness of response is coded as if the child's response had been wrong. Evidence for guessing can be the child's statement to that effect or little evidence of thought in generating a quick response with a number that may or may not be close to the actual answer.

Given Number. This is an inappropriate strategy in which the child responds with one of the numbers given in the problem. The interviewer must determine that the child generated the answer by choosing one of the numbers given in the problem rather than by miscounting when using some other strategy or recalling an incorrect number fact. When subjects respond with a number given in the problem and report that they have "guessed," Given Number is coded.

Wrong Operation. This category is coded when the child adds the two numbers given in a subtraction problem or subtracts the numbers in an addition problem. The strategy is only determined to be inappropriate; no attempt is made to identify the particular inappropriate addition or subtraction strategy used.

Errors

Just as certain strategies are not independent of the model used, e.g., Separating From requires the use of physical models, certain errors are not independent of the strategy used. The strategies Guess, Given Number, and Wrong Operation can be considered errors in that they lead to an incorrect answer. Since they do, however, describe the actions children take upon their representations of the problem, they are considered strategies, although inappropriate. Wrong Operation involves misinterpretation of the problem, while Guess and Given Number involve a lack of interpretation of the problem.

Procedural errors also were coded. For the following categories an appropriate strategy was chosen, but an incorrect answer resulted. Incorrect recall of a number fact is one procedural error; this is recorded by coding the Number Fact strategy along with an incorrect answer.

Miscount. This error is coded if the child counts incorrectly when using a strategy involving concrete representation, for example, failing to count an object or counting an object twice. Miscounting also occurs in conjunction with counting strategies when a number is omitted or when the entry or exit numbers in the counting sequence are incorrectly included or excluded.

<u>Forget</u>. This procedural error occurs when a child forgets part of the problem data and an error results from the use of this incorrect information. If it is suspected that this error has occurred, it is necessary to question the child after completion of the problem, asking questions such as, "How many stamps did Leroy have to begin with?", or "How many stamps did his mother give him?", or "What were the numbers in the problem?".

Several errors can occur on the same problem. In some such instances both errors are coded and in others only one is coded. When two procedural errors occur in the same problem both are coded, for example, a child forgets one of the numbers in the problem and then miscounts in determining the solution. Multiple errors such as Miscount and Forget typically occur in conjunction with an appropriate strategy. When the Wrong Operation error occurs it is deemed to be the major cause of the wrong answer, and even if the child also forgets the problem data or miscounts, those procedural errors are not coded once Wrong Operation is coded. Other errors are not coded along with Given Number or Guess, since these uniquely determine the child's error.

In addition to the above errors others were anticipated. Pilot work suggested that for problems that required more than ten objects some children might mistakenly respond with "10" once all fingers had been used. Responses such as "three plus five is six" were also anticipated since several subjects in the pilot studies had based answers on the successor of the last number given in the problem. Because of these potential errors and because little was known about the types of errors kindergartners might exhibit, the interviewers attempted to document any errors that did not clearly fit predetermined categories in order to develop other error categories in a post hoc fashion. Additional categories of errors and strategies resulting from the analysis of subjects' responses are given in Chapter IV.

Coder Agreement

The three coders each had been trained to use the coding categories described previously. Prior to data collection intra-coder and inter-coder agreement were measured using video-taped segments of interviews with primary-grade children. In all instances the level of intra-coder and inter-coder agreement was greater than .90. Although the coder training and agreement assessment were done using verbal problems, the pilot studies suggested that children's strategies, errors, and modeling procedures for abstract problems could be coded using the same categories as had been developed for verbal problems. Thus, no additional measure of coder agreement was done for coding repsonses to abstract problems.

The aforementioned empirical procedures guided the data collection. The methods used to analyze the data and the results of those analyses are described in Chapter IV.

Chapter IV

DATA ANALYSES AND RESULTS

The preceding chapters have described the background, design, and data collection procedures for the study. ~~The procedures selected were consistent~~ with the study's purpose of describing and comparing kindergarten and first-grade children's performance on addition and subtraction problems presented in abstract and verbal problem contexts. This chapter describes the analyses of the data and the results of those analyses. The initial section of the chapter presents preliminary results that are consequences of the administration of the tasks and initial analysis of the data. The principal data analyses and results follow, and are discussed in reference to the research questions set forth in Chapter III.

Preliminary Analyses

The entire set of abstract and verbal problems was administered to each subject. Although several subjects guessed frequently, no subject appeared uneasy or upset by the interviews. Consequently, analyses were carried out using a complete set of data for each subject.

Errors in Data Collection

Three errors occurred in the data collection, but none was deemed to have had any substantial influence on children's performance. The first error consisted of four incorrect assignments of number triples to verbal problems at the kindergarten level. For example, (1,3,4) was used in place of (1,4,5). Since number triples were randomly ~~assigned to problems and each number~~ triple occurred often with each type of problem, this error was assumed not to have altered the subjects' processes for solving the problems.

In three instances kindergarten subjects were given an abstract missing addend problem that involved the incorrect form of a given number triple. For example, 2+__=5 was used instead of 3+__=5. This was assumed not to have had any systematic influence on the subject's solution process.

The third data collection error involved administration of problems originally intended for one kindergarten subject to another subject and vice versa. Since tasks were randomly assigned to subjects, this one deviation from the original assignment of problem decks to subjects was not expected to influence the results.

Clerical Procedures Applied to Coded Data

The original coding sheets used by the interviewers were reviewed by the experimenter for errors or inconsistencies in coding. Several were found and were resolved immediately after the data collection took place. Information from the original coding sheets was transferred to another sheet for computer processing. To determine whether data were reliably transferred, a random sample of five percent of these sheets was checked for agreement with the original sheets. No discrepancies were found, so it was assumed that the transfer of data was accurate. Checks for coding inconsistencies, e.g., a correct response with the Wrong Operation strategy, were performed during initial data analysis; after these were complete it was assumed that the only errors in the data were those caused by the inevitable subjectivity of the coders' interpretations during the interviews.

Coding Category Changes

The interviewers' anecdotal accounts were used to clarify any ambiguous responses. Four additional inappropriate strategy categories and two new error categories have been defined to subsume those instances for which none of the previous coding categories was appropriate. Also, no subject used the Matching strategy, so this category was not used in the data analyses.

Model Both Sets. This faulty separating strategy for subtraction problems involves construction of two sets and removal of one of them rather than construction of a set and removal of a subset of it. For example, when solving 8-3=__, the child forms a set of eight objects, forms a set of three objects, separates or removes the set of three, and concludes that the answer is eight after counting the remaining set of eight objects. When this strategy is used a child appropriately attempts to concretely represent the problem and remove some objects but is unable to correctly carry out the Separate From strategy because the action performed on the sets was incorrect.

All Cubes Used. In this strategy faulty modeling or representation of the numbers or sets in the problem leads to an incorrect answer. It is demonstrated when a child inappropriately uses all of the available manipulatives when representing an addition or subtraction problem. For example, if twenty cubes are available when solving 8-3=__ or 3+8=__ the child might form sets of eight cubes and three cubes, but derive the answer by counting the remaining nine cubes. Another instance of this strategy occurs when a child forms only one set and counts all of the remaining cubes to determine the answer. Although it is possible to obtain a correct answer by chance using this strategy, this did not occur.

Inappropriate Fact. Subjects occasionally attempted to use a known addition or subtraction fact on a problem for which that fact was inappropriate. An example of this would be a response of "3+3" for the problem 2+__=6. In this instance the child knows that 3+3 is 6, but is unable to recall a number fact involving the required addend (two).

<u>Add On/Given Number</u>. Occasionally, subjects had difficulty determining the set of objects to count to determine the answer after carrying out an Adding On strategy. The Add On/Given Number strategy involved initially using an Adding On strategy to model the action or relationship in the problem, but answering with the larger number given in the problem. This reflected a difficulty in identifying the answer set among the manipulatives or fingers that were present after the required number had been adjoined to the original set (Lindvall & Ibarra, Note 7). For example, when solving 2+__=8, after six objects had been adjoined to the original two, the subject had difficulty identifying the six within the set of eight objects. This strategy does not involve counting of the adjoined set, therefore, it is considered distinct from the Adding On strategy.

<u>Two additional error categories</u>. Two types of errors were identified which could not be classified in the original coding categories. The Ten Fingers error resulted from subjects' inability to use fingers to model

the data in problems in which the sum or minuend was greater than ten. Subjects would occasionally use an appropriate strategy but give "ten" as the answer simply because they did not have enough fingers to model the entire problem. This error could also occur if only one hand were used and "five" were given as the answer.

The other new error category, Configuration, was similar to the error resulting from the Add On/Given Number strategy in that it also resulted from subjects' inability to determine the answer from the final configuration of manipulatives or fingers. The Configuration error was coded whenever the subject's incorrect response was based on some perceptually compelling aspect of the final configuration of the objects. For example, Configuration was coded if, when solving 2+__=9, after mistakenly adding on eight instead of seven fingers, the child realized the mistake, removed the one finger, but then focused on the one just removed, giving "1" as the answer.

Comparison of Data from the Two Schools

The data from schools A and B were compared with respect to the frequency of correct repsonses and the frequency of occurrence of various solution strategies and errors. Subjects from school A correctly solved the addition problems more often than did subjects from school B; p-values were lower by approximately .10 in school B. The p-values for subtraction problems were comparable for the two schools.

In most instances solution strategies were used with similar frequencies by subjects from the two schools. Inspection of the data from the two schools indicated that subjects in school A exhibited concrete representation strategies (using fingers or cubes) more frequently than subjects in school B. Errors involving the use of the wrong operation on subtraction problems occurred more often in school A, whereas, guessing and responding with one of the numbers given in the problem occurred more often in school B. Subjects in school B attempted to use number facts more than subjects from school A. A brief comparison of the strategy frequency and correct response data for the two schools is given in Appendix H.

The preceding differences in strategy use and correct responses by subjects from the two schools are minimal. Thus, data from the two schools are combined in subsequent analyses.

Children's Performance on Abstract and Verbal Problems

Question 1

What strategies do children in grades K and 1 use to solve addition and subtraction verbal problems?

The data on strategies used for solving the verbal problems are analyzed descriptively. Results for addition and subtraction problems are summarized separately. The first-grade results are compared to those of Carpenter and Moser (1981).

Addition problems. Table 4 presents the percentage of use of the most frequently used strategies on verbal addition problems. In both grades strategy use was similar for the Join and Combine problems. At the kindergarten level Counting All, which involves concrete representation of the problem, was the predominant strategy. At the first-grade level concrete representation predominated when manipulatives were available, and when no manipulatives were available, mental strategies (primarily recall of number facts) were used most frequently for problems with small numbers and counting strategies for those with larger numbers.

The preceding first-grade findings were consistent with the results of Carpenter and Moser (1981). Table 5 presents the strategies used on verbal problems in the present study and the Carpenter and Moser study. These are summarized according to three qualitatively different levels of abstraction: concrete representation, counting, and mental strategies. Since the present study

Table 4

Percentage Use of Strategies on Verbal Addition Problems

		Grade K				Grade 1			
		Join		Combine		Join		Combine	
		S	L	S	L	S	L	S	L
Counting All	C	34*	54	44	62	41	56	30	67
	N					11	7	7	15
Subitize	C	4	0	6	0	7	0	11	0
	N					11	0	19	7
Counting On From Smaller	C	6	6	4	8	7	19	4	15
	N					11	22	15	19
Counting On From Larger	C	2	0	6	4	7	19	7	7
	N					15	22	11	22
Number Fact	C	28	8	18	4	30	4	41	4
	N					37	11	41	7
Derived Fact	C	4	0	0	0	0	4	0	4
	N					0	0	0	4
Guess	C	10	12	10	10	0	0	4	0
	N					7	15	4	7
Given Number	C	10	12	4	0	0	4	0	0
	N					0	7	0	15

* Columns do not sum to 100 because seldom-used strategies are not included.

S - small numbers (sums less than 6, grade K)
(sums 6 through 9, grade 1)

L - larger numbers (sums 6 through 9, grade K)
(sums 11 through 15, grade 1)

C - Manipulatives (cubes) available

N - No manipulatives available

Table 5

Percentage of Use of Three Categories of Strategies by First-graders in the Present Study and in Carpenter and Moser (1981)

	Concrete Representation				Counting Strategies				Mental Strategies			
	Small Numbers		Larger Numbers		Small Numbers		Larger Numbers		Small Numbers		Larger Numbers	
Problem Type	C	N	C	N	C	N	C	N	C	N	C	N
Join												
Carpenter and Moser (January)	56	38	53	17	14	19	20	32	20	20	5	6
Present Study	48	22	56	7	14	38	26	44	30	37	8	11
Carpenter and Moser (May)	37	20	47	16	20	28	33	51	35	43	13	14
Combine												
Carpenter and Moser (January)	53	36	49	19	13	18	23	25	19	23	5	7
Present Study	41	26	67	22	11	26	22	41	41	41	8	11
Carpenter and Moser (May)	36	24	47	15	23	29	34	54	31	36	13	8
Separate												
Carpenter and Moser (January)	63	40	69	18	5	5	4	11	14	13	3	2
Present Study	44	33	78	33	7	19	8	29	34	33	4	11
Carpenter and Moser (May)	57	44	65	24	9	16	16	25	26	27	9	15
Join/Change Unknown												
Carpenter and Moser (January)	46	29	44	11	16	16	12	26	18	19	6	3
Present Study	44	11	44	7	11	30	22	33	30	37	11	4
Carpenter and Moser (May)	32	21	52	15	20	26	18	44	36	35	13	12

C - Manipulatives (cubes) available
N - No manipulatives available

occurred in March one would expect its results to fall between those of the January and May interviews of the Carpenter and Moser study.

Even though the Join and Combine problems elicit similar solution strategies, children who use Counting All might use different modeling procedures for these two problems (Heller, Note 21). The two types of modeling that have been observed are construction of two distinct sets, joining of these sets, and counting of the union set starting from the number one on Combine problems; and construction of an initial set, one-by-one incrementing of that set by the amount to be joined, and counting of the resulting set starting with the number one for Join problems. In other studies (Lindvall & Ibarra, Note 7; Riley & Greeno, Note 4) pauses built into the reading of the verbal problems may have elicited different modeling procedures for Join and Combine problems. In the present study problems were read in their entirety without pausing to require the subject to model the most recently read portion of the problem.

Table 6 presents the frequency with which various modeling procedures were used. Modeling procedures for these problems can be categorized as consistent with the structure of the problem (use of one set for Join and two sets for Combine), inconsistent (use of two sets for Join

Table 6

Number of Subjects Exhibiting Four Types of Modeling Procedures Used With Counting All

Modeling Procedure	Grade K	Grade 1
Consistent	4	3
Inconsistent	1	1
Uniform	21	9
Mixed	1	3

Consistent - Use one set for Join, two sets for Combine

Inconsistent - Use two sets for Join, one set for Combine

Uniform - Use one set (or two sets) for both problems

Mixed - Use the Consistent procedure for problems at one number size level and Uniform for problems at the other number size level and one for Combine), uniform (same modeling procedure for both problems), and mixed (e.g., use of a consistent procedure for small number problems and inconsistent or uniform for larger number problems). The majority of subjects in both grades used a uniform modeling procedure for both types of problems. Several subjects in each grade used one set for Join and two for Combine problems, but one subject in each grade modeled the problems in the opposite way. The fact that inconsistencies and mixed modeling procedures occurred and that most subjects modeled both problems the same way provides further evidence that children's solution processes for verbal addition problems are not influenced by the structure of the problem.

Subtraction problems. Table 7 presents strategies used for verbal subtraction problems. In both grades the strategies used on the two types of subtraction problems, Separate and Join/Change Unknown, are those that most directly model the action described in the problem. The concrete representation and counting strategies used on Separate problems are in marked contrast to those used on the Join/Change Unknown problems. For the former, the subtractive strategies Separate From and Counting Down From predominated; one kindergarten subject's use of Counting Up From Given is the only instance in which an

Table 7

Percentage Use of Strategies on Verbal Subtraction Problems

		Separate				Join/Change Unknown			
		Small Difference		Large Difference		Small Difference		Large Difference	
		S	L	S	L	S	L	S	L
Grade K									
Separate From		48*	46	34	48	0	0	0	0
Counting Down From		0	2	2	4	0	0	0	0
Adding On		0	0	0	0	24	36	36	34
Counting Up From Given		0	0	2	0	6	10	6	2
Number Fact		22	8	32	8	40	6	10	0
Derived Fact		0	0	0	0	0	0	2	2
Guess		12	22	12	16	14	22	8	24
Given Number		6	8	6	6	4	8	6	12
Wrong Operation		4	4	2	4	10	12	14	10
Add On/ Given Number		0	0	0	0	0	6	6	6
Grade 1									
Separate From	C	44	78	52	56	0	0	4	0
	N	33	33	41	22	0	0	0	0

(continued)

Table 7 (continued)

		Separate				Join/Change Unknown			
		Small Difference		Large Difference		Small Difference		Large Difference	
		S	L	S	L	S	L	S	L
Counting Down From	C	7	4	11	22	0	0	0	0
	N	15	22	11	37	0	0	0	0
Counting Down To	C	0	4	0	0	0	0	0	0
	N	4	7	0	0	0	0	0	0
Adding On	C	0	0	0	0	44	44	48	52
	N	0	0	0	0	11	7	22	7
Counting Up From Given	C	0	0	0	0	11	22	3	19
	N	0	0	0	0	30	33	15	30
Number Fact	C	30	4	22	7	26	11	30	4
	N	33	7	41	19	33	4	30	7
Derived Fact	C	4	0	0	0	4	0	0	0
	N	0	4	0	0	4	0	0	0
Guess	C	0	0	0	4	0	4	4	0
	N	11	7	4	7	7	15	4	19
Given Number	C	0	0	0	0	0	0	0	0
	N	0	4	0	0	0	11	7	4
Wrong Operation	C	11	7	7	7	11	4	7	4
	N	4	4	4	7	4	7	15	19

* Columns do not sum to 100% because seldom-used strategies are omitted.

S - small numbers (sums less than 6, grade K)
(sums 6 through 9, grade 1)

L - larger numbers (sums 6 through 9, grade K)
(sums 11 through 15, grade 1)

C - Manipulatives (cubes) available

N - No manipulatives available

additive strategy was used for a Separate problem. The subtractive strategies reflect subset removal or its counting analog, while the additive strategies, Adding On and Counting Up From Given, used nearly universally for the Join/Change Unknown problems, reflect the joining rather than the separating process.

At the kindergarten level concrete representation strategies were the predominant ones. However, when the numbers in the problem were small, number facts were used frequently on the Separate problems with large differences (5-1=__, 4-1=__, and 5-2=__) and for the Join/Change Unknown problems with small differences (3+__=4, 4+__=5, 3+__=5). This is not surprising since these problems afford an opportunity to use addition and subtraction facts involving one and two as well as the opportunity to easily hide a counting strategy or to use it subconsciously.

At the first-grade level, when manipulatives were available, concrete representation strategies predominated. For small number problems mental strategies were used more than counting strategies, and the reverse was true when the problems contained larger numbers. When no objects were available, counting and mental strategies were often used as frequently or more frequently than concrete representation. These results closely paralleled the mid-year findings of the Carpenter and Moser (1981) study (see Table 5).

Summary. For verbal addition problems with different problem structures, children's solution processes did not reflect those differences in structure. However, for verbal subtraction problems, children's strategies overwhelmingly mirrored the structure of the problem.

Thus, data from the present study provide strong support for the descriptions by Carpenter and Moser (1981) of the relationship between problem structure and children's strategies for solving Separate, and Join/Change Unknown problems. The consistency with which the frequencies of use of strategies in the present study fell within the range of frequencies of the mid- and end-of-year interviews of the Carpenter and Moser study indicates that, in spite of a difference in the mathematics text series used, first-graders in the two studies used essentially similar strategies for solving verbal problems.

Question 2

> What strategies do children in grades K and 1 use to solve abstract addition and subtraction problems?

Addition problems. Table 8 presents the kindergarten and first-grade subjects' average percentage use of strategies on the two parallel abstract addition problems. At the kindergarten level, concrete representation strategies were predominant, counting strategies were used infrequently, and mental strategies were used frequently only on problems with small numbers. The first-grade subjects used Counting All and Subitize much less frequently, relying primarily on recall of number facts when the numbers were small and counting strategies when the problems contained larger numbers.

Subtraction problems. Table 9 presents the strategies used by kindergartners and first-graders on the four types of abstract subtraction problems. The strategies used were those that reflected the structure of the problem. Across both grades, with the exception of one subject, additive strategies (Adding On and Counting Up From Given) were never used for the problems based on a-b=__. Similarly, with the exception of small difference missing addend problems with larger numbers, subtractive strategies (Separate From and Counting Down From) were used infrequently for the missing addend problems. Thus, although a few subjects in each grade occasionally used subtractive concrete representation strategies for the abstract missing addend problems,

Table 8

Percentage Use of Strategies on Abstract Addition Problems

		Grade K		Grade 1	
		Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Counting All	C	35 *	53	11	31
	N			2	9
Subitize	C	11	4	4	0
	N			4	0
Counting On From Smaller	C	2	1	7	17
	N			9	26
Counting On From Larger	C	4	8	17	35
	N			26	24
Number Fact	C	21	3	56	6
	N			56	13
Derived Fact	C	4	0	2	4
	N			2	2
Guess	C	15	19	4	6
	N			0	13
Given Number	C	3	7	0	2
	N			0	2
Wrong Operation	C	0	3	0	0
	N			2	0

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

Table 9

Percentage Use of Strategies on Abstract Subtraction Problems

	a-b=__				a+__=c			
	Small Difference		Large Difference		Small Difference		Large Difference	
	S	L	S	L	S	L	S	L
Grade K								
Separate From	48*	54	38	42	0	0	2	4
Counting Down From	2	0	4	10	0	0	0	0
Adding On	0	0	0	0	32	32	26	30
Counting Up From Given	0	0	0	0	8	10	6	4
Number Fact	2	0	20	6	14	2	6	0
Derived Fact	0	2	0	0	0	6	2	0
Guess	22	26	16	28	22	18	12	22
Given Number	10	10	8	8	4	6	8	4
Wrong Operation	4	2	4	2	14	12	22	22
Add On/ Given Number	0	0	0	0	4	8	6	6

(continued)

Table 9 (continued)

		a-b=__				a+__=c			
		Small Difference		Large Difference		Small Difference		Large Difference	
		S	L	S	L	S	L	S	L
Grade 1									
Separate From	C	41	70	26	56	0	22	4	11
	N	37	30	33	30	0	0	0	0
Counting Down From	C	0	15	0	7	0	0	0	0
	N	4	22	19	41	0	0	4	0
Adding On	C	0	0	0	0	15	15	22	44
	N	0	0	0	0	15	7	11	7
Counting Up From Given	C	0	0	0	0	19	41	11	15
	N	4	0	0	0	19	26	15	26
Number Fact	C	44	7	63	22	48	4	33	7
	N	33	4	48	7	41	11	37	11
Derived Fact	C	4	0	0	4	0	0	4	0
	N	11	4	0	0	4	4	0	0
Guess	C	4	4	7	7	4	11	4	7
	N	4	11	0	11	0	33	7	26
Given Number	C	0	0	0	0	4	4	7	4
	N	0	0	0	0	0	0	0	0
Wrong Operation	C	4	0	4	4	7	4	11	4
	N	4	0	0	0	15	7	15	15

* Columns do not sum to 100 because seldom-used strategies are not included.

S - small numbers (sums less than 6, grade K)
(sums 6 through 9, grade 1)

L - larger numbers (sums 6 through 9, grade K)
(sums 11 through 15, grade 1)

C - Manipulatives (cubes) available

N - No manipulatives available

children's concrete representation and counting strategies for abstract subtraction problems nearly always reflected the structure of the problems.

Question 3

> Within each of grades K and 1 are there differences between children's ability to solve addition and subtraction problems presented in verbal problem context and their ability to solve corresponding problems presented in an abstract context?

Children's ability to solve a problem is measured by two criteria, correctness of the answer and ability to use an appropriate solution strategy. Each subject's responses are classified dichotomously on correctness, with No Attempt being considered an incorrect response. Responses are also classified dichotomously on the use of an appropriate strategy. Strategies that yield correct answers but are uncodable (ambiguous) are classified as appropriate strategies, and those uncodable strategies that yield wrong answers are classified as inappropriate. Tables 10 and 11 present a summary of subjects' percentage of correct answers and use of appropriate strategies on verbal and abstract problems containing small and larger numbers.

To test for differences in difficulty between

Table 10

Percentage of Correct Responses

Problem Type	Percent Correct: Small Numbers	Percent Correct: Larger Numbers
Grade K		
Join	70	50
Combine	72	66
Abstract (a+b=__)	80	62
Separate (small difference)	56	40
Abstract (a-b=__, small difference)	50	42
Separate (large difference)	66	50
Abstract (a-b=__, large difference)	60	46
Join/Change Unknown (small difference)	70	40
Abstract (a+__=c, small difference)	52	40
Join/Change Unknown (small difference)	52	30
Abstract (a+__=c, large difference)	38	22
Grade 1		
Join	80	57
Combine	78	69
Abstract (a+b=__)	89	63
Separate (small difference)	63	57
Abstract (a-b=__, small difference)	81	61
Separate (large difference)	81	52
Abstract (a-b=__, large difference)	89	46
Join/Change Unknown (small difference)	80	39
Abstract (a+__=c, small difference)	76	46
Join/Change Unknown (large difference)	69	41
Abstract (a+__=c, large difference)	67	48

Table 11

Percentage of Use of Appropriate Strategies

Problem Type	Percentage Use of Appropriate Strategy	
	Small Numbers	Larger Numbers
Grade K		
Join	80	70
Combine	80	80
Abstract (a+b=__)	81	70
Separate (small difference)	72	60
Abstract (a-b=__, small difference)	54	56
Separate (large difference)	72	64
Abstract (a-b=__, large difference)	64	58
Join/Change Unknown (small difference)	70	52
Abstract (a+__=c, small difference)	54	50
Join/Change Unknown (large difference)	60	44
Abstract (a+__=c, large difference)	42	38
Grade 1		
Join	94	78
Combine	93	85
Abstract (a+b=__)	97	85
Separate (small difference)	85	83
Abstract (a-b=__, small difference)	89	83
Separate (large difference)	89	81
Abstract (a-b=__, large difference)	94	85
Join/Change Unknown (small difference)	85	63
Abstract (a+__=c, small difference)	81	65
Join/Change Unknown (large difference)	76	61
Abstract (a+__-c, large difference)	74	63

corresponding verbal and abstract problems as measured by correctness of response and by ability to apply appropriate strategies, the Cochran Q Test (Marascuilo & McSweeney, 1977) is used. At the kindergarten level four contrasts of interest are those comparing performance on: the Join problem and the two abstract addition items, a+b=__; the Combine problem and the two abstract addition items; the two Separate problems (small and large difference) and the corresponding abstract subtraction items, a-b=__; and the two Join/Change Unknown problems (small and large difference) and the corresponding abstract missing addend items, a+__=c. Each of these four contrasts is tested for problems with small numbers (sums less than 6) and for problems with larger numbers (sums 6 through 9). These eight contrasts are computed using correctness of response as the measure of difficulty and also with use of an appropriate strategy as the measure of difficulty.

Table 12 presents the results of the Cochran Q Test for the kindergarten data. When correctness of response is used as the measure of difficulty, none of the contrasts is significant at the .05 level. Hence, there are no significant differences between kindergartners' ability to solve (as measured by correctness) each of the four types of verbal problems (Join, Combine, Separate,

Table 12

Verbal--Abstract Problem Contrasts, Grade K

Criterion	Contrast		Significance
Correctness of Response	C1-7,8 *	= -.09	ns (p>.05)
	C2-7,8	= -.07	ns
	C3,4-9,10	= -.06	ns
	C5,6-9,10	= .16	ns
	C13-19,20	= -.13	ns
	C14-19,20	= .07	ns
	C15,16-21,22	= .03	ns
	C17,18-23,24	= .04	ns
Use of Appropriate Strategy	C1-7,8	= -.02	ns
	C2-7,8	= -.02	ns
	C3,4-9,10	= .13	ns
	C5,6-11,12	= .06	ns
	C13-19,20	= .00	ns
	C14-19,20	= .10	ns
	C15,16-21,22	= .05	ns
	C17,18-23,24	= .04	ns

* Numbers refer to the problem types below:

Small Number Problems	Larger Number Problems
1 - Join	13
2 - Combine	14
3 - Separate (small difference)	15
4 - Separate (large difference)	16
5 - Join/Change Unknown(small difference)	17
6 - Join/Change Unknown(large difference)	18
7 - a+b=__	19
8 - a+b=__	20
9 - a-b=__ (small difference)	21
10- a-b=__ (large difference)	22
11- a+__=c (small difference)	23
12- a+__=c (large difference)	24

and Join/Change Unknown) and corresponding abstract problems. Similarly, when ability to apply an appropriate strategy is used as the measure of difficulty, none of the contrasts is significant at the .05 level. Thus, there are no significant differences between kindergartners' ability to use an appropriate strategy to solve each of the four verbal addition and subtraction problems and corresponding abstract problems.

At the first-grade level the preceding four contrasts at each number size level are expanded to six. Kindergarten-level subtraction problems with small and large differences are very similar when sums are less than six, for example, 5-2=__ and 5-3=__ are not markedly different. Consequently, the small difference and large difference problems are aggregated. At the first-grade level none of the sums used are less than six, so it is reasonable to test for differences in difficulty between Separate problems with small differences and corresponding abstract problems, as well as between Separate problems with large differences and their corresponding abstract problems. Similarly, the one contrast involving Join/Change Unknown problems at the kindergarten level is broken into two at the first-grade level. Thus, there are six contrasts at each of the number size levels for the first-grade data.

Results of the comparisons for the first-grade data are presented in Table 13. When correctness of response is used as the measure of difficulty, none of the contrasts is significant at the .05 level. First-graders' ability to obtain correct answers on each of the six types of verbal problems is not significantly different from their ability to correctly solve a corresponding problem presented in abstract number sentence context. When ability to use an appropriate strategy is the measure of difficulty, none of the contrasts is significant at the .05 level. Hence, there are no significant differences in first-graders' ability to apply appropriate strategies to any of the six types of verbal addition and subtraction problems and their number sentence counterparts.

Thus, verbal and abstract problems were of equal difficulty for kindergartners, both when correctness of response was the criterion for subjects' ability to solve the problems and when use of an appropriate strategy was the criterion. Likewise, verbal and abstract problems were of equal difficulty for first-graders when each of the preceding criteria were used.

Question 4

Within each of grades K and 1 are there differences between the strategies children use to

Table 13

Verbal--Abstract Contrasts, Grade 1

Criterion	Contrast			Significance
Correctness of Response	C1-7,8 *	=	-.09	ns (p>.05)
	C2-7,8	=	-.11	ns
	C3-9	=	-.19	ns
	C4-10	=	-.07	ns
	C5-11	=	.04	ns
	C6-12	=	.02	ns
	C13-19,20	=	-.06	ns
	C14-19,20	=	.05	ns
	C15-21	=	-.04	ns
	C16-22	=	.05	ns
	C17-23	=	-.07	ns
	C18-24	=	-.07	ns
Use of Appropriate Strategy	C1-7,8	=	-.03	ns
	C2-7,8	=	-.05	ns
	C3-9	=	-.04	ns
	C4-10	=	-.06	ns
	C5-11	=	.04	ns
	C6-12	=	.02	ns
	C13-19,20	=	-.06	ns
	C14-19,20	=	.01	ns
	C15-21	=	.00	ns
	C16-22	=	-.04	ns
	C17-23	=	-.02	ns
	C18-24	=	-.02	ns

* The numbers in these contrasts refer to the problems described in Table 12.

solve verbal addition and subtraction problems and those used for corresponding abstract problems?

Data pertinent to this question are analyzed descriptively. For each of grades K and 1 the strategies subjects used on the abstract and the two verbal addition problems are compared. For subtraction problems two independent descriptive analyses are performed. In each grade the strategies used on the two (small and large difference) Separate problems and their abstract counterparts are compared and a similar comparison is made for strategies used on the two Join/Change Unknown problems and corresponding abstract problems. Percentage differences less than 15% are considered too small to be of any educational significance and are not dealt with in the following presentation of results.

Individual strategies are categorized in six qualitatively different classes: concrete representation strategies that directly model the structure of the problem, strategies that involve concrete representation but do not reflect the structure of the problem (Adding On used on a Separate problem, for example), counting reflecting problem structure, counting not reflecting problem structure (for example, Counting Down From for 3+__=9), mental, and inappropriate strategies. The complete categorization of strategies is given in Table

Table 14

Strategies Categorized in Six Levels

Strategy	Category
Counting All	CR
Subitize	CR
Counting On From Smaller	C
Counting On From Larger	C-NRS
Separate From	CR for a-b=__ CR-NRS for a+__=c
Separate To	CR-NRS
Adding On	CR for a+__=c CR-NRS for a-b=__
Counting Down From	C for a-b=__ C-NRS for a+__=c
Counting Down To	C-NRS
Counting Up From Given	C for a+__=c C-NRS for a-b=__
Derived Fact	M
Number Fact	M
Inappropriate Fact	I
No Attempt	I
Uncodable	I if answer is correct, omitted if answer is incorrect

(continued)

Table 14 (continued)

Strategy	Category
Given Number	I
Wrong Operation	I
Models Both Sets	I
All Cubes Used	I
Add On/Given Number	I

CR -- Concrete Representation
(reflecting problem structure)

CR-NRS -- Concrete Representation
Not Reflecting problem Structure

C -- Counting
(reflecting problem structure)

C-NRS -- Counting
Not Reflecting problem Structure

M -- Mental

I -- Inappropriate

14. In the following discussion strategies are treated individually within the class of inappropriate strategies.

Addition problems. Performance on Join and Combine problems is similar enough in both grades to warrant collapsing the data for these two verbal problems. Tables 15 and 16 present the percentage of strategies in the preceding levels used on verbal and abstract addition problems in grades K and 1, respectively. At the kindergarten level the strategies used for problems in the two contexts are similar. Concrete representation strategies (Counting All and Subitizing) are predominant in both contexts and the frequencies of use of concrete representation, counting, and mental strategies are nearly the same for the two contexts.

In grade 1 the abstract problems elicited fewer concrete representation strategies. Even when manipulatives were available for abstract problems with larger numbers, more subjects used counting strategies than concrete representation. This contrasts with subjects' less frequent use of counting and more frequent use of concrete representation on verbal problems for which manipulatives were available.

Subtraction problems. Tables 17 and 18 present the percentage of subjects in grades K and 1 employing

Table 15

Kindergartners' Percentage Use of Strategies on Verbal and Abstract Addition Problems

Strategy Category	Join and Combine		Abstract (a+b=__)	
	Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Concrete Representation	44 *	58	46	57
Concrete Representation Not Reflecting Structure	-	-	-	-
Counting	5	7	2	1
Counting Not Reflecting Structure	4	2	4	8
Mental	25	6	25	3
Inappropriate				
Guess	10	11	15	19
Given Number	7	6	3	7
Wrong Operation	0	3	0	3

* Columns do not sum to 100 because seldom-used strategies are not included.

Table 16

First-graders' Percentage Use of Strategies on Verbal and Abstract Addition Problems

Strategy Category		Join and Combine Small Numbers	Join and Combine Larger Numbers	Abstract (a+b=__) Small Numbers	Abstract (a+b=__) Larger Numbers
Concrete Representation	C	45 *	62	15	31
	N	24	15	3	5
Concrete Representation Not Reflecting Structure	N	-	-	-	-
	N	-	-	-	-
Counting	C	6	17	7	17
	N	13	21	9	26
Counting Not Reflecting Structure	C	7	13	17	35
	N	13	22	26	24
Mental	C	36	8	58	10
	N	39	11	58	15
Inappropriate					
Guess	C	2	0	4	6
	N	6	11	0	13
Given Number	C	0	2	0	2
	N	0	11	0	2
Wrong Operation	C	0	0	0	0
	N	0	0	0	0

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

Table 17

Kindergartners' Percentage Use of Strategies on Small Difference Verbal and Abstract Subtraction Problems Based on a-b=__

	Separate Small Difference		Abstract (a-b=__) Small Difference	
Strategy Category	Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Concrete Representation	48 *	46	48	54
Concrete Representation Not Reflecting Structure	0	0	0	0
Counting	0	2	2	0
Counting Not Reflecting Structure	0	0	0	0
Mental	22	8	2	2
Inappropriate				
Guess	12	22	22	26
Given Number	6	8	10	10
Wrong Operation	4	4	4	2

* Columns do not sum to 100 because seldom-used strategies are not included.

Table 18

First-graders' Percentage Use of Strategies on Small Difference Verbal and Abstract Subtraction Problems Based on a-b=__

		Separate Small Difference		Abstract (a-b=__) Small Difference	
Strategy Category		Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Concrete Representation	C	44 *	78	41	70
	N	33	33	37	30
Concrete Representation Not Reflecting Structure	C	0	0	0	0
	N	0	0	0	4
Counting	C	7	4	0	15
	N	15	22	4	22
Counting Not Reflecting Structure	C	0	4	0	0
	N	4	7	4	7
Mental	C	34	4	48	7
	N	33	11	44	8
Inappropriate					
Guess	C	0	0	4	4
	N	11	7	4	11
Wrong Operation	C	11	7	4	0
	N	4	4	4	0

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

various strategies on small difference Separate problems and the corresponding abstract problem a-b=__. Tables 19 and 20 present similar percentages of use of strategies on parallel large difference problems. From these tables it is clear that kindergartners used similar strategies to solve these verbal and abstract problems. The incidence of use of strategies that reflected the structure of the problem is similar for verbal and abstract problems. Concrete representation strategies reflecting problem structure predominated.

At the first-grade level the strategies employed for Separate and corresponding abstract subtraction problems similarly reflect the structure of the problems. For both abstract and verbal problems the concrete representation and counting strategies not reflecting problem structure were either not used or used rarely. The strategies employed on small difference abstract and verbal problems were used with similar frequencies. When manipulatives were available, mental strategies were used more often and concrete representation less often on large difference abstract problems than on corresponding verbal problems.

Tables 21 and 22 present strategies most frequently used by subjects in grades K and 1 on small difference Join/Change Unknown problems and the corresponding

Table 19

Kindergartners' Percentage Use of Strategies on Large Difference Verbal and Abstract Subtraction Problems Based on a-b=__

	Separate Large Difference		Abstract (a-b=__) Large Difference	
Strategy Category	Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Concrete Representation	34 *	48	38	42
Concrete Representation Not Reflecting Structure	0	0	0	0
Counting	2	4	4	10
Counting Not Reflecting Structure	2	0	0	0
Mental	32	8	20	6
Inappropriate				
Guess	12	16	16	28
Given Number	6	6	8	8
Wrong Operation	2	4	4	2

*. Columns do not sum to 100 because seldom-used strategies are not included.

Table 20

First-graders' Percentage Use of Strategies on Large Difference Verbal and Abstract Subtraction Problems Based on a-b=__

Strategy Category		Separate Large Difference Small Numbers	Separate Large Difference Larger Numbers	Abstract (a-b=__) Large Difference Small Numbers	Abstract (a-b=__) Large Difference Larger Numbers
Concrete Representation	C	52 *	56	26	56
	N	41	22	33	30
Concrete Representation Not Reflecting Structure	C	0	0	0	0
	N	0	0	0	0
Counting	C	11	22	0	7
	N	11	37	19	41
Counting Not Reflecting Structure	C	0	0	0	0
	N	0	0	0	4
Mental	C	22	7	63	26
	N	41	19	48	7
Inappropriate					
Guess	C	0	4	7	7
	N	4	7	0	11
Wrong Operation	C	7	7	4	4
	N	4	7	0	0

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

Table 21

Kindergartners' Percentage Use of Strategies on Small Difference Verbal and Abstract Missing Addend Problems

Strategy Category	Join/Change Unknown Small Difference: Small Numbers	Join/Change Unknown Small Difference: Larger Numbers	Abstract (a+__=c) Small Difference: Small Numbers	Abstract (a+__=c) Small Difference: Larger Numbers
Concrete Representation	24 *	36	32	32
Concrete Representation Not Reflecting Structure	0	0	0	0
Counting	6	10	8	10
Counting Not Reflecting Structure	0	0	0	0
Mental	40	6	14	8
Inappropriate				
Guess	14	22	22	18
Given Number	4	8	4	6
Wrong Operation	10	12	14	12
Add On/ Given Number	0	6	4	8

* Columns do not sum to 100 because seldom-used strategies are not included.

Table 22

First-graders' Percentage Use of Stratgies on Small Difference Verbal and Abstract Missing Addend Problems

		Join/Change Unknown Small Difference		Abstract(a+__=c) Small Difference	
Strategy Category		Small Numbers	Larger Numbers	Small Numbers	Larger Numbers
Concrete Representation	C	44 *	44	15	15
	N	11	7	15	7
Concrete Representation Not Reflecting Structure	C	0	0	0	22
	N	0	0	0	0
Counting	C	11	22	19	41
	N	30	33	19	26
Counting Not Reflecting Structure	C	0	0	0	0
	N	0	0	0	0
Mental	C	30	11	48	4
	N	37	4	45	15
Inappropriate					
Guess	C	0	4	4	11
	N	7	15	0	33
Given Number	C	0	0	4	4
	N	0	11	0	0
Wrong Operation	C	11	4	7	4
	N	4	7	15	7

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

abstract problem a+__=c. Tables 23 and 24 present similar percentages of use of strategies on parallel large difference problems. As with the previous subtraction problems, it is clear that kindergartners used the same strategies to solve these verbal and abstract missing addend problems. The concrete representation and counting strategies used reflect problem structure in both contexts.

At the first-grade level, except for small difference abstract problems with larger numbers (see Table 22), the concrete representation and counting strategies used on Join/Change Unknown problems and abstract problems such as a+__=c reflect the structure of the problems. The principal difference in the frequency with which appropriate strategies were used on these abstract and verbal missing addend problems is less use of concrete representation on abstract than verbal problems when manipulatives were available. This decreased use of concrete representation was offset by increased use of number facts when problems contained small numbers and by increased use of counting strategies when problems contained larger numbers.

Use of the wrong operation by first-graders is comparable for problems in the two contexts. However, guessing occurred more frequently on small difference

Table 23

Kindergartners' Percentage Use of Strategies on Large Difference Verbal and Abstract Missing Addend Problems

Strategy Category	Join/Change Unknown Large Difference: Small Numbers	Join/Change Unknown Large Difference: Larger Numbers	Abstract(a+__=c) Large Difference: Small Numbers	Abstract(a+__=c) Large Difference: Larger Numbers
Concrete Representation	36 *	34	26	30
Concrete Representation Not Reflecting Structure	2	6	2	4
Counting	6	2	6	4
Counting Not Reflecting Structure	0	0	0	0
Mental	12	2	8	0
Inappropriate				
Guess	8	24	12	22
Given Number	6	12	8	4
Wrong Operation	14	10	22	22
Add On/ Given Number	6	6	6	6

* Columns do not sum to 100 because seldom-used strategies are not included.

Table 24

First-graders' Percentage Use of Strategies on Large Difference Verbal and Abstract Missing Addend Problems

Strategy Category		Join/Change Unknown Large Difference: Small Numbers	Join/Change Unknown Large Difference: Larger Numbers	Abstract (a+__=c) Large Difference: Small Numbers	Abstract (a+__=c) Large Difference: Larger Numbers
Concrete Representation	C	48 *	52	22	44
	N	22	7	11	7
Concrete Representation Not Reflecting Structure	C	4	0	8	11
	N	0	0	0	0
Counting	C	3	19	11	15
	N	15	30	15	26
Counting Not Reflecting Structure	C	0	0	0	0
	N	0	0	4	0
Mental	C	30	4	37	7
	N	30	7	37	11
Inappropriate					
Guess	C	4	0	4	7
	N	4	19	7	26
Given Number	C	0	0	7	4
	N	7	4	0	0
Wrong Operation	C	7	4	11	4
	N	15	19	15	15

* Columns do not sum to 100 because seldom-used strategies are not included.
C - Manipulatives (cubes) available
N - No manipulatives available

abstract missing addend problems than on small difference Join/Change Unknown problems.

Summary. At both grade levels the types of strategies elicited by addition and subtraction problems clearly reflect problem structure for both abstract and verbal contexts. At the kindergarten level the frequencies with which these strategies were used are similar for abstract and verbal addition problems as well as subtraction problems. For both addition and subtraction, first-graders display differences in the frequencies with which certain strategies were used on verbal and abstract problems. The principal difference was that concrete representation sometimes occurred less on abstract than verbal problems, usually when manipulatives were available. Thus, Question 4 can be answered negatively at the kindergarten level and affirmatively at the first-grade level.

Question 5

> Are kindergarten and first-grade children's strategies for solving verbal or abstract problems different for problems with small numbers than for problems with larger numbers?

The descriptive analysis for this question is based on the data in Tables 15 through 24 concerning the percentage of use of six classes of strategies. At both grade levels concrete representation and counting strategies not reflecting problem structure were used infrequently enough to preclude the existence of any meaningful differences in the use of these strategies on small and larger number problems in either the verbal or the abstract context.

The principal difference in strategy use on small and larger number problems is less frequent use of mental strategies on larger number problems than on those with smaller numbers. For addition problems this difference occurred on both abstract and verbal problems in both grades. For subtraction problems first-graders employed fewer mental strategies on the larger number problems in both contexts, but kindergartners only did so on large difference Separate and small difference Join/Change Unknown problems. At the first-grade level the decrease in mental strategies on larger number problems was often accompanied by increased use of either counting or concrete representation strategies.

For small and large difference abstract missing addend problems and large difference Join/Change Unknown problems the decrease in mental strategies on larger number problems was also accompanied by an increase in guessing. Thus, primarily at the first-grade level, there was a higher incidence of more primitive strategies on larger number problems than on those with smaller numbers.

Question 6

> Do first-graders who use counting strategies to solve verbal and/or abstract subtraction problems use strategies which mirror problem structure or strategies which reflect attention to the efficiency of alternative counting procedures?

Data for this question are generated from the strategies used by first-graders on the pairs of related subtraction problems with small and large differences, e.g., 8-6=__ and 8-2=__ or 6+__=8 and 2+__=8. Tables 25 and 26 classify individuals according to their strategies on the small and large difference abstract and verbal problems based on a-b=__ and a+__=c, respectively.

The most efficient counting strategies for these problems are Counting Up From Given or Counting Down To on the small difference problems and Counting Down From on large difference problems, since these minimize the number of counts used. To determine whether individuals used counting strategies that reflected attention to the efficiency of the counting process it is necessary to test whether the proportions of subjects using Counting Up From Given or Counting Down To differ for the large

Table 25

Number of First-grade Subjects Exhibiting Counting Strategies for Pairs of Abstract and Pairs of Verbal Problems Based on a-b=__.

a) Abstract - small numbers

Small Difference Problem	Large Difference Problem: UG,DT	DF	Other
UG,DT	0	1	0
DF	0	0	1
Other	0	4	48

b) Abstract -larger numbers

Small Difference Problem	Large Difference Problem: UG,DT	DF	Other
UG,DT	1	1	0
DF	0	7	3
Other	0	5	37

(continued)

Table 25 (continued)

c) Separate - small numbers

		Large Difference Problem		
		UG,DT	DF	Other
Small Difference Problem	UG,DT	0	0	1
	DF	0	3	3
	Other	0	3	44

d) Separate - larger numbers

		Large Difference Problem		
		UG,DT	DF	Other
Small Difference Problem	UG,DT	0	2	1
	DF	0	6	1
	Other	0	8	36

UG,DT - Counting Up From Given or Counting Down To

DF - Counting Down From

Other - Any strategy other than UG,DT or DF

Table 26

Number of First-grade Subjects Exhibiting Counting Strategies for Pairs of Abstract and Pairs of Verbal Problems Based on a+__=c.

a) Abstract - small numbers

Small Difference Problem	Large Difference Problem: UG, DT	DF	Other
UG, DT	5	1	4
DF	0	0	0
Other	2	0	42

b) Abstract - larger numbers

Small Difference Problem	Large Difference Problem: UG, DT	DF	Other
UG, DT	8	0	10
DF	0	0	0
Other	3	0	33

(continued)

Table 26 (continued)

c) Separate - small numbers

		Large Difference Problem		
		UG,DT	DF	Other
Small Difference Problem	UG,DT	4	0	7
	DF	0	0	0
	Other	1	0	42

d) Separate - larger numbers

		Large Difference Problem		
		UG,DT	DF	Other
Small Difference Problem	UG,DT	8	0	7
	DF	0	0	0
	Other	5	0	34

UG,DT - Counting Up From Given or Counting Down To

DF - Counting Down From

Other - Any strategy other than UG,DT or DF and small difference problems. Likewise, it is necessary to determine whether the proportions of subjects using Counting Down From differ on large difference and small difference problems. Since 3 X 3 tables are involved, the appropriate test is the Stuart test for the equality of correlated marginal probabilities (Marascuilo and McSweeney, 1977).

Since the Stuart test is performed for eight contingency tables, the level of significance chosen is .01. This ensures that an overall level less than .10 is maintained across use of this test. The critical value for significance of the contrasts involving the differences in proportions of strategy use on small and large difference problems is 2.447.

Table 27 gives the value of the two contrasts of interest for each of the contingency tables. For problems based on a+b=__ and for those based on a+__=c, the proportions of use of Counting Up From Given and Counting Down To on small and large difference problems do not differ significantly for abstract or for verbal problems with small and larger numbers. Likewise, no differences in the proportions of use of Counting Down From on small and large difference problems occur under any of the conditions. Thus, although older children may do so, there is no evidence that first-graders base their

Table 27

Contrasts Involving First-graders' Counting Strategies for Two Types of Subtraction Problems

Source		Contrast	Significance
Table 25	a)	* C1 = .019	ns (p>.01)
		C2 = -.074	ns
	b)	C1 = .018	ns
		C2 = -.056	ns
	c)	C1 = .019	ns
		C2 = .000	ns
	d)	C1 = .056	ns
		C2 = -.166	ns
Table 26	a)	C1 = .055	ns
		C2 = -.019	ns
	b)	C1 = .129	ns
		C2 = .000	ns
	c)	C1 = .111	ns
		C2 = .000	ns
	d)	C1 = .037	ns
		C2 = .000	ns

*C1= p.1-p1. = P(UG,DT used for small difference problem) -P(UG,DT used for large difference problem)

C2= p.2-p2. = P(DF used for small difference problem) -P(DF used for large difference problem) counting strategies on the efficiency of the counting process.

Inspection of Tables 25 and 26 indicates that subjects who used counting strategies for both small and large difference problems nearly always used Counting Down From for both problems based on a-b=__. Counting Up From Given was nearly always used for both problems based on a+__=c. This indicates that counting strategies mirrored the semantic structure of the subtraction problems regardless of the size of the difference between the numbers.

<u>Differences in Performance in Grades K and 1</u>

<u>Question 7</u>

> Are there differences in the level of abstraction of kindergartners' and first-graders' strategies?

The descriptive analysis for this question is based on a comparison of kindergartners' and first-graders' frequency of use of strategies categorized within qualitatively different and increasingly abstract strategy levels: inappropriate, concrete representation, counting, and mental. Table 28 presents the percentage of total strategies in each of these strategy levels for the two grades. Since the use of concrete representation strategies could be influenced by the presence of

Table 28

Use of Four Categories of Strategies as Percent of Total Strategy Use

	Strategy Category							
	Inappropriate		Concrete Representation		Counting		Mental	
	Grade							
Problem Type	K	1*	K	1	K	1	K	1
Join	27	8	46	52	7	20	20	19
Combine	22	6	56	54	11	17	11	25
Abstract (a+b=__)	27	6	51	23	7	38	14	34
Separate (small difference)	37	13	47	61	1	8	15	19
Abstract (a-b=__, small difference)	47	8	51	56	1	8	2	28
Separate (large difference)	35	13	41	54	4	17	20	15
Abstract (a-b=__, large difference)	40	11	40	41	7	4	13	45
Join/Change Unknown (small difference)	39	19	30	44	8	17	23	21
Abstract (a+__=c, small difference)	48	19	32	26	9	30	11	26
Join/Change Unknown (large difference)	50	20	39	52	4	11	7	17
Abstract (a+__=c, large difference)	60	23	31	43	5	13	4	22

* First-grade subjects with manipulatives available

manipulatives, the grade K data are compared only to the data from first-graders who had objects available.

The incidence of use of the more abstract counting and mental strategies was higher for grade 1 than grade K. The differences between grade K and grade 1 in the percentage of total strategies falling into the concrete representation category are mixed; on some items first-graders used concrete representation more than kindergartners, and in others the percentages are nearly equal. This is primarily a result of first-graders' use of fewer inappropriate strategies. First-graders used concrete representation much less frequently than kindergartners did only on abstract addition problems.

For most problems, a greater percentage of first-graders' than kindergartners' strategies consists of the more abstract counting and mental strategies. At the kindergarten level strategies are primarily inappropriate or concretely-based, and at the first-grade level they are concretely-based or more abstract. Thus, Question 7 can be answered affirmatively.

<u>Question 8</u>

Are there differences in the flexibility with which kindergartners and first-graders choose among alternative strategies reflecting and not reflecting problem structure?

The data in Tables 15 through 24 indicate that other than on addition problems at the first-grade level, children in both grades seldom used concrete representation and counting strategies not reflecting problem structure. Especially on subtraction problems such strategies were used so seldomly as to preclude drawing any meaningful conclusions about differences between subjects in the two grades. Furthermore, when subjects' strategies for pairs of problems with identical number sizes and identical underlying number sentences (e.g., small and large difference Separate problems) are examined, it is evident that the most common occurrence on such pairs of problems is the use of strategies from the same category (e.g., both mental or both concrete representation reflecting problem structure). On only 2% of the 600 such pairs of problems at the kindergarten level and 4% of the 648 pairs of problems at the first-grade level did subjects use one concrete representation or counting strategy directly reflecting problem structure and one not reflecting problem structure. Due to such limited occurrences, Question 8 must be answered negatively.

Question 9

Are kindergartners' errors of interpretation qualitatively different from those exhibited by first-graders?

Data for this question are analyzed descriptively. With one exception, the errors of interpretation exhibited by the kindergartners are the same as those exhibited by the first-graders. Use of a correct but inappropriate number fact is the only error that occurred exclusively at the kindergarten level.

Question 10

Do various types of errors in solving addition and subtraction problems occur with differing relative frequencies for kindergarten and first-grade children?

The description of errors includes a discussion of several errors of interpretation that are also considered inappropriate strategies and were discussed to some extent in previous questions. The present discussion focuses on comparing the ocurrence of these errors (strategies) for the two grade levels. Table 29 gives the frequency of errors at the two grade levels on verbal and abstract problems. These frequencies are compared for problems with small numbers and for problems with larger numbers. Thus, the comparison between grades is based on problems with different number sizes.

Procedural Errors. The frequency of occurrence of procedural errors in grades K and 1 varies depending on

Table 29

Frequency of Errors

	Verbal Small Numbers		Abstract Small Numbers		Verbal Larger Numbers		Abstract Larger Numbers		TOTAL *	
Grade:	K	1	K	1	K	1	K	1	K	1
PROCEDURAL ERRORS										
Miscount	5	24	4	11	30	56	27	49	66	140
Forget	20	6	7	2	13	5	7	2	47	15
Incorrect Fact	7	10	3	8	4	8	1	8	15	34
ERRORS OF INTERPRETATION										
Superficial										
Guess	33	12	51	10	53	21	66	40	203	83
Given Number	19	2	18	3	25	12	21	5	83	22
Inappropriate Fact	1	0	3	0	1	0	0	0	5	0
No Attempt	4	2	6	4	9	17	5	16	24	39
Wrong Operation	15	17	22	17	18	16	22	9	77	59
Incorrect Modeling										
Model Both Sets	0	0	4	1	2	0	5	1	11	2
All Cubes Used	1	6	1	0	0	8	0	0	2	14
Identification of Answer										
Add On/ Given Number	3	0	5	0	6	3	7	0	21	3
Ten Fingers	0	0	0	0	1	9	2	10	3	19
Configuration	0	0	0	0	0	0	2	2	2	2

* Total Responses: Grade K -- 1200 (N=50)
Grade 1 -- 1296 (N=54)

the type of error. Miscounting occurred more frequently at the first-grade level. This is undoubtedly due to first-graders' more frequent use of strategies involving counting and the inclusion of problems with larger numbers which offer more opportunities to miscount. At each grade level miscounting occurred more frequently on problems with larger numbers. Forgetting the problem data occurred less frequently among first-grade subjects. Use of an incorrect number fact occurred twice as often among first-graders as kindergartners, but again, this was likely due to first-graders' more frequent use of number facts.

Errors of interpretation. Four categories of interpretation errors are used. Superficial errors include guessing, responding with a number given in the problem, use of an inappropriate number fact, and making no attempt to solve the problem. Use of the wrong operation comprises a second category, and two types of incorrect modeling, modeling both sets in subtraction and attempting to use all available manipulatives, comprise a third category of interpretation errors. Incorrect identification of the answer set is the fourth category, and includes the error inherent in the Add On/Given Number strategy, errors involving the final configuration of manipulatives or fingers, and the errors caused by limitation to ten fingers.

Superficial errors of interpretation occurred less frequently at the first grade level than at the kindergarten level. Overall, guessing was used less than half as often by first-graders than kindergartners, but when first-grade subjects are classified according to those with and without manipulatives available, there are marked differences in the frequency of guessing. Those first-graders with manipulatives available guessed about one-tenth as often as the kindergartners, but the first-graders who had no manipulatives guessed more than half as often as the kindergarten subjects. Responding with a number given in the problem occurred about one-fourth as often among first-grade subjects as among kindergartners. The Inappropriate Fact error occurred only at the kindergarten level. Making no attempt to solve the problem occurred with approximately equal frequency in the two grades, but when subjects are again classified by the availability of manipulatives, making no attempt occurred only once when objects were available but thirty-eight times when no objects were available. Thus, if the kindergarten subjects are compared with the first-graders who had objects available, no attempt occurred much less frequently at the first-grade level. In contrast, the 27 first-graders who had no objects available made no attempt to solve the problem about three times as often per subject as the kindergartners who did have objects available.

In contrast to superficial errors, which involve a <u>lack</u> of interpretation of the problem, use of the wrong operation (an <u>incorrect</u> interpretation of the problem) occurred only slightly less often at the first-grade level. In both grades this error occurred most often on verbal and abstract missing addend problems.

Incorrect modeling errors, which involve incorrect <u>representation</u> of the problem, occurred infrequently at each grade level. First-graders seldom modeled both sets on subtraction problems, and few subjects in either grade attempted to use all of the available manipulatives (although one aberrant first-grader used this strategy on twelve problems).

Results of the comparison of identification of the answer errors across the two grades are mixed. The Add On/Given Number error occurred much less frequently among first-graders, but due to larger numbers being used in the first-grade problems, the Ten Fingers error occurred more frequently in grade one. Errors based on the final configuration of objects or fingers occurred infrequently in both grades.

<u>Summary</u>. Although procedural errors occurred

frequently in both grades, there are differences in the frequency of forgetting the problem data and miscounting across the two grades. On errors of interpretation there are greater differences between kindergartners and first-graders. Many of the errors of interpretation occurred less frequently among the first-grade subjects than among the kindergartners. First-graders failed to interpret problems (used superficial solutions) less than the kindergarten subjects did, and they seldom incorrectly modeled the subtraction problems of the form a-b=__. However, first-graders incorrectly interpreted problems (used the wrong operation) and committed errors involving identification of the answer set nearly as frequently as kindergartners. In contrast to the kindergarten subjects, the first-graders were willing to attempt to solve virtually all of the problems if manipulatives were available. When no objects were available, however, first-graders made no attempt to solve problems more often than the kindergartners (all of whom had objects available).

Although some types of errors occurred relatively infrequently overall, the number of subjects exhibiting those errors is often substantial. For example, use of the wrong operation occurred on less than seven percent of the total responses at the kindergarten level, yet 52%

of the kindergarten subjects used the wrong operation on at least one problem. Less than five percent of the first-grade responses involved use of the wrong operation, but 35% of the first-graders used the wrong operation at least once. The percentage of subjects exhibiting modeling errors at least once declines across the two grade levels, with only 4% of the first-graders as compared to 18% of the kindergartners exhibiting such errors. The number of subjects who experienced difficulty in identification of the answer set is nearly the same for the two grades (30% in grade K and 28% in grade 1). Thus, just as most errors occurred less frequently at the first-grade level, the number of subjects exhibiting those errors also is lower at the first-grade level.

Individual Differences in Solution Processes

Question 11

> What individual differences occur among kindergarten and first-grade children in their ability to solve and their strategies for solving verbal and abstract addition and subtraction problems, i.e., within each grade level can interpretable clusters of children be formed according to the types of problems they can solve and the types of strategies they employ?

Four cluster analyses are used to group subjects within each of the two grade levels on two dimensions, the types of problems they could solve and the strategies they used. In order to cluster subjects by the types of problems they could solve (as measured by use of an appropriate strategy), it is necessary to determine the freqency with which each subject used appropriate strategies on six types of problems. Appendix H gives these frequencies for verbal addition problems, verbal subtraction problems based on a - b=__, verbal subtraction problems based on a+__=c, abstract addition problems, abstract subtraction problems based on a-b=__, and abstract subtraction problems based on a+__=c. Since there are two problems of each type within each of the two number size levels, each subject's frequency of use of an appropriate strategy on a given problem type can range from 0 through 4.

Appendix I gives the frequency with which each subject used various solution strategies across the verbal and abstract problems. Since kindergartners used some counting and concrete representation strategies infrequently, the original set of nineteen strategies is collapsed to fifteen by combining the frequencies for Counting All and Subitizing; Counting From Smaller and Counting From Larger; Counting Up From Given, Counting

Down From, and Counting Down To; and Number Fact and Derived Fact. First-graders' strategies are similarly collapsed. Each subject solved twenty-four problems, so the frequencies in Appendix I can range from 0 to 24.

The data in Appendices H and I are the basis for the clusterings of subjects on the dimensions involving the types of strategies they used and the types of problems on which they used an appropriate strategy. Complete link hierarchical clustering was performed using the Clustering Research Program (Baker, Note 22). The similarity measure for a pair of subjects was the sum of the squares of differences of standardized scores over problem types in the one analysis, and over strategy types in the other. Elbows in the graph of the diameters (differences in similarity values for the grouping) at each level of clustering were used to determine the iteration at which to interpret the clustering.

When kindergarten subjects are clustered by strategy use, five clusters of subjects are formed. Table 30 lists the subjects included in each cluster. Cluster KS1 includes five subjects who frequently used counting and mental strategies. Cluster KS2 includes subjects who often attempted to use mental strategies and/or guessing to solve the problems. The more frequent incidence of inappropriate strategies distinguishes this cluster from

Table 30

Clusterings of Kindergarten Subjects

Cluster	Subjects
Clustering by Types of Strategies Used	
KS1	(1, 17, 18, 36, 40)
KS2	(3, 44, 35, 7, 12, 26, 23, 48, 42, 38, 46)
KS3	(2, 43, 6, 32, 45, 30, 47, 15, 24, 21, 28, 37, 8, 5, 49, 50, 19, 16, 20, 31, 25, 13, 22, 4, 29, 41, 9, 11, 14, 39)
KS4	(34)
KS5	(10, 27)
Clustering by Use of Appropriate Strategies on Six Types of Problems	
KA1	(1, 4, 9, 2, 32, 36, 43, 18, 24, 28, 45 40, 6, 50, 15, 21, 47, 37, 25)
KA2	(5, 30, 39, 14, 49)
KA3	(8, 11, 31)
KA4	(29, 33)
KA5	(16, 17, 41, 20)
KA6	(3, 48)
KA7	(23)
KA8	(7, 38, 35, 46, 12, 44, 42)
KA9	(26, 27)
KA10	(10, 22)
KA11	(13, 19, 34)

KS1. Subjects in cluster KS3 comprise the bulk of the kindergarten sample (31 subjects). This cluster is distinguished by its frequent use of concrete representation strategies, primarily Counting All, Separating From, and Adding On. Cluster KS4 consists of one subject who frequently made no attempt to solve the problem and cluster KS5 consists of two subjects whose strategies included concrete representation and making no attempt. The analysis, therefore, yields three main clusters: subjects (KS3) who relied on concrete representation strategies and possibly used a number of inappropriate strategies, subjects who used more abstract counting and mental strategies (KS1), and the subjects in KS2 who attempted to solve problems abstractly (often unsuccessfully).

The clustering of kindergartners along the dimension of problem types on which appropriate strategies were used yields the eleven clusters in Table 30. Cluster KA1 includes subjects who solved (used an appropriate strategy on) nearly all problems. Subjects in KA2 solved nearly all problems except the abstract missing addend problems. KA3 subjects experienced difficulty primarily on abstract additon and abstract missing addend problems. Cluster KA4 includes subjects whose primary difficutly occurred on Separate problems. Subjects in KA5 solved verbal problems and abstract addition problems, but few others. KA6 includes subjects who solved some verbal problems but few or no abstract problems. A single subject who succeeded only on Separate problems comprises cluster KA7. KA8 consists of seven subjects who used few, if any, appropriate strategies. The subjects in cluster KA9 solved no verbal problems but a few abstract problems. KA10 includes two subjects who solved only addition problems (verbal and abstract). The final cluster, KA11, includes subjects who did not solve missing addend problems in either context.

Clusters KS3 and KA1 each represent substantial portions of the kindergarten sample. Comparison of the membership across clusters can be used to further distinguish among the membership of the large clusters in either dimension. For example, the subjects of cluster KS3, who primarily used concrete representation strategies, are members of seven of the categories of the other clustering (all but KA6, KA7, KA8, and KA9). Similarly, the members of KA1, who used appropriate strategies on nearly all problems, are split among clusters KS1 and KS3. Thus, a large number of subjects who are homogeneous in terms of the strategies they used differ widely in the types of problems on which they used appropriate strategies. Similarly, a large group of subjects who appear homogeneous in terms of the problems they could solve, can be discriminated among by the types of strategies they used.

The clustering of first-graders according to the strategies they used yields twelve clusters. Table 31 lists the subjects included in each cluster. Cluster FS1 includes subjects who used concrete representation strategies, some counting strategies, number facts, and only a few inappropriate strategies. The second cluster, FS2, is somewhat similar to the first, with subjects employing concrete representation, using number facts, and exhibiting counting strategies primarily on addition problems. Cluster FS3 includes four subjects who made no attempt on approximately one-fourth of the problems, usually those with larger numbers, and often attempted to recall number facts. This cluster used some concrete representation strategies and often guessed or used other inappropriate strategies. FS4 consisted of a single subject who used Counting All, guessed, and made no attempt to solve nine of the problems. Cluster FS5 is characterized by a high incidence of guessing. When not guessing, the three subjects in FS5 used different strategies, one using primarily concrete representation, another using concrete representation and mental strategies, and the other using counting and mental

Table 31

Clusterings of First-grade Subjects

Cluster	Subjects
Clustering by Types of Strategies Used	
FS1	(1, 53, 51, 37, 38, 47, 7, 16, 3, 8, 12, 28)
FS2	(22, 36, 41)
FS3	(5, 11, 48, 42)
FS4	(39)
FS5	(6, 43, 44)
FS6	(2, 50, 52, 20, 49, 30, 35, 46, 10, 19)
FS7	(17, 23, 24, 25)
FS8	(4, 14)
FS9	(9, 33, 34, 40)
FS10	(13, 54, 15, 21, 26, 27, 31, 29)
FS11	(18, 45)
FS12	(32)
Clustering by Use of Appropriate Strategies on Six Types of Problems	
FA1	(1, 2, 3, 10, 12, 19, 21, 46, 47, 50 52, 53, 54, 41, 7, 8, 22, 15, 36, 35 13, 20, 49, 51, 24, 38, 31, 33, 34, 40 9, 29, 16, 28, 30, 37, 26, 23, 27, 45)
FA2	(4, 14, 17)
FA3	(5, 18, 25, 42)
FA4	(6, 11, 48)
FA5	(32, 44, 39)
FA6	(43)

strategies. Cluster FS6 includes ten subjects who recalled number facts as their primary strategy, who used counting strategies on both addition and subtraction problems and also used number facts, or who used number facts along with some counting and concrete representation strategies. The subjects in cluster FS7 each used several uncodable strategies. Their other strategies were less homogeneous, with one using concrete representation, counting, and mental strategies, two using counting and mental strategies, and another using primarily mental strategies. Subjects in cluster FS8 are distinguished by their use of the wrong operation on nearly half of the problems. The four subjects in cluster FS9 primarily used counting strategies for the addition problems, concrete representation for subtraction problems of the form a-b=__, and concrete representation or counting and inappropriate strategies for problems based on a+__=c. Frequent use of counting strategies characterizes the eight subjects in cluster FS10. The two subjects in cluster FS11 often used guessing and responding with a number given in the problem when problems involved larger numbers. One subject who frequently used incorrect modeling procedures (the All Cubes Used and Model Both Sets strategies) comprises cluster FS12.

When first-graders are clustered according to the frequency with which they applied appropriate strategies to different types of problems, the six clusters in Table 31 are formed. Cluster FA1 consists of forty of the fifty-four first-grade subjects. This large cluster includes subjects who used appropriate strategies on all problems, some who used an occasional inappropriate strategy, and some who used fewer appropriate strategies on missing addend problems than on the other problems. The subjects in cluster FA2 were able to solve all addition problems, abstract problems based on a-b=__ but not the Separate problems, and some or none of the missing addend problems. Cluster FA3 contains subjects who solved all problems based on a-b=__, some of the addition problems, and only a few of the missing addend problems. Use of appropriate strategies only on about half of the problems distinguishes cluster FA4. The subjects in cluster FA5 solved some abstract problems but seldom solved verbal problems. The final cluster, FA6, contains one subject who could solve verbal but not abstract problems.

When the memberships of the two first-grade clusterings are compared the results match, in part, the same comparison at the kindergarten level. Subjects in the large cluster, FA1, who are homogeneous in terms of using appropriate strategies for all or nearly all problems, are heterogeneous in terms of the strategies they used. These subjects fall into seven clusters on the dimension of strategies used. However, among the subjects in the large clusters (FS1, FS6, and FS10) who are somewhat homogeneous in terms of strategy use, there is no heterogeneity of problem types to which appropriate strategies were applied. All subjects from clusters FS1, FS6, and FS10 are members of cluster FA1. This contrasts with the clusterings at the kindergarten level in which a large group of subjects who are homogeneous on strategy use are heterogeneous according to the problem types they could solve.

Summary

Verbal and abstract problems were of equal difficulty for subjects in both grades. Although kindergartners used essentially the same strategies to solve verbal and abstract problems, first-graders exhibited less frequent use of concrete representation strategies on abstract than verbal problems. In both grades the strategies used for subtraction problems closely reflected problem structure in both contexts.

First-graders' strategies entailed a greater degree of abstraction than those of kindergartners, however, there were no differences in the flexibility with which subjects in the two grades used strategies reflecting and not reflecting problem structure. Subjects in the two grades committed essentially the same types of errors, although the frequency of occurrence of most errors was lower at the first-grade level. At both grade levels there were a variety of individual differences in the types of strategies subjects used and the types of problems they could solve.

Chapter V

DISCUSSION

The purpose of this study was to describe and compare kindergarten and first-grade children's strategies for solving certain addition and subtraction problems. The questions of interest focused on differences between children's performance on problems presented in abstract and verbal problem contexts, differences between the performance of kindergartners and first-graders on such problems, and individual differences in the addition and subtraction problem solving performance of children at these two grade levels. The verbal problems used in the study were Join and Combine addition problems and Separate and Join/Change Unknown subtraction problems. The corresponding abstract problems were of the forms a+b=__, a-b=__, and a+__=c. Much of the analysis of the data from the study was descriptive and was intended to provide a basis for better understanding the solution processes children use on addition and subtraction problems.

This chapter interprets specific results of the study and uses those results to characterize children's performance in somewhat broader terms. The chapter presents instructional implications of the results as well as implications for future research. The limitations of the study are also discussed; these serve to qualify the interpretation of the results as well as to provide a basis for future research directions.

Interpretation of Results

Discussion of the results of the study is done in two parts. The results pertaining to kindergarten and first-grade children's ability to solve abstract and verbal problems are discussed first. This is followed by interpretation of the results concerning children's solution processes.

Children's Ability to Solve Abstract and Verbal Problems

The results clearly indicate that many kindergarten children as well as first-graders can solve verbal and abstract problems based on a+b=__, a-b=__, and a+__=c. At both grade levels addition problems (both verbal and abstract) were the easiest for children to solve. Three of the four subtraction problem types were roughly comparable in difficulty, with the exception being that missing addend problems with the difference greater than the given addend were more difficult than the others.

By clustering subjects according to the problem types they could solve, the present study extends the results of previous studies concerned with item difficulty. Although in nearly all cases a majority of the kindergartners applied appropriate strategies to each problem type, individual subjects varied as to the problems they could solve. Some kindergartners could solve only addition problems (both verbal and abstract); some could solve addition problems and the subtraction problems based on a-b=__; some could solve addition problems, subtraction problems based on a-b=__, and verbal missing addend problems; some could solve only verbal addition problems, the subtraction problems based on a-b=__, and only verbal missing addend problems; some could solve all types; and a few could solve only the Separate problems or all but the Separate problems. The distinctions among first-graders according to the problem types solved are less numerous, possibly reflecting the influence of instruction on some of the problems. In addition to the first-graders who could solve all the problem types, one group of subjects solved addition problems, abstract but not the verbal problems based on a -b=__, and only a few missing addend problems; and another group solved all problems based on a-b=__, some addition problems, and few missing addend problems.

The preceding results suggest that children within each of these grade levels are in no way homogeneous in terms of the types of problems they can solve. It is important for teachers and curriculum developers to recognize that all of the problems used in this study are readily solvable by many children in grades K and 1. They must also be aware of the differences among individuals; the types of problems a kindergartner can solve may vary greatly, while somewhat less variability may be present at the first-grade level. Teachers should take advantage of opportunities to individually assess their students' ability to solve a variety of addition and subtraction problems.

Previous studies have provided little data pertinent to comparing children's performance on verbal problems and corresponding abstract or symbolic problems. Verbal problems are often thought to be more difficult than abstract problems. However, an important result of the present study is that kindergarten and first-grade children are able to solve verbal problems just as easily as abstract problems. Without direct instruction on either abstract or verbal problems, more than half of the kindergarters were able to apply appropriate strategies on nearly all problem types. Also, without instruction on verbal problems, between 60 and 95% of the first-graders applied appropriate strategies to these problems. Thus, at these grade levels, verbal addition and subtraction problems (in which the child is not required to read the problem) cannot be considered too difficult to be included in the curriculum.

The lack of differences in difficulty between verbal and abstract problems suggests that verbal problems are a potential vehicle for initial work relating to the operations of addition and subtraction. Much of the emphasis in most first-grade mathematics curricula is placed on teaching children to become fluent with symbolically represented problems and recall of basic addition and subtraction facts. Results from the present study indicate that first-graders who have had substantial instruction on abstract problems and little or no instruction on verbal problems performed equally well on problems in these two contexts. The obvious question that derives from this finding concerns the influence of the ability to solve problems in one context on the ability to solve problems in the other. One could argue that instruction on symbolic problems facilitated first-graders' performance on verbal problems. This is undoubtedly true, to some extent. However, it appears that kindergartners can solve verbal problems at least as well as abstract problems. In fact, overall, kindergartners applied appropriate strategies more often to verbal problems than to abstract problems. This suggests that some young children may solve verbal problems without necessarily solving abstract problems or learning addition and subtraction facts.

A reasonable conclusion concerning children's acquisition of the capability to solve verbal and abstract problems is that three situations may occur. For some children experiences with processes such as joining, separating, comparing and equalizing may provide the basis for them to be able to solve verbal problems (using concrete representation strategies) prior to the time at which they can solve abstract problems containing no cues such as "getting" and "giving away." The clusterings by problem type solved yield eight such subjects; seven kindergartners and one first-grader solved verbal problems but had difficulty with abstract problems.

For other children abstract problems may be easier to solve initially because they do no contain the verbiage that verbal problems do. Four kindergartners and six first-graders exhibited difficulty in solving certain verbal problems but no such difficulty on corresponding abstract problems. Two of these kindergartners were identified by their teacher as being in special language programs because of demonstrated language deficiencies.

For a third group of children the capabilities to solve abstract and verbal problems may develop concurrently (either interrelatedly or independently). A large number of subjects at both grade levels exhibited equal facility with verbal and abstract problems. While longitudinal data would be necessary to determine whether these children developed the capabilities to solve verbal and abstract problems simultaneously, it is plausible that some of these children developed these capabiltities concurrently. In particular, children who understand that an abstract problem can be associated with each verbal problem may develop the ability to solve abstract and verbal problems of a particular type at the same time.

The preceding conclusions suggest that teachers may need to introduce the addition and subtraction operations to some children via verbal problems, problems they can already solve. Other students may have difficulty understanding the prose in verbal problems and may profit more from working with problems similar to the abstract problems used with the kindergarten subjects. A wise course of action would be for teachers to use verbal problems whenever possible to supplement or supplant the more limited emphasis on joining and separating of sets of objects currently used to provide initial experiences with addition and subtraction. Such an emphasis on verbal problems would provide both the opportunity for some students to build their understanding of addition and subtraction on familiar problems and processes, as well as providing an opportunity to learn processes for solving verbal problems.

Children's Strategies for Verbal and Abstract Problems

The concrete representation and counting strategies kindergartners and first-graders used for both verbal and abstract problems are those that directly model the action or relationship in the problem. Young children clearly have independent conceptions of the various types of subtraction problems.

Even though children in both grades exhibited the same set of strategies for verbal and abstract problems, one difference repeatedly emerges concerning the frequencies with which strategies were used. On problems that were most familiar to the first-grade subjects (those based on a+b=__ and a-b=__), they used the more abstract counting and mental strategies more frequently on problems presented in the abstract context than the verbal context and concrete representation more frequently on verbal than abstract problems. When problems were less familiar to first-graders (a+__=c), they used concrete representation more frequently on the verbal than the abstract problems and sometimes used guessing more frequently on abstract than verbal problems.

The preceding results can be viewed from two perspectives. Since abstract problems in some instances elicit more sophisticated counting and mental strategies, it can be argued that the structure of verbal problems is so compelling that children use concrete representation even though they are capable of utilizing more abstract strategies. However, when children are just beginning to learn to solve certain problem types, the structure of verbal problems may be sufficiently salient to enable children to solve problems which they otherwise might not have solved. This suggests that verbal and abstract problems may serve different but complementary purposes in instruction on addition and subtraction. Verbal problems might be used best to introduce various problem types to children. For example, subtraction problems can entail several different problem structures (additive, comparative, or subtractive). Verbal problems appear to be the most appropriate ones for effectively introducing these to children. Abstract problems may be most effectively used when the goal of instruction is to encourage children to develop or use more abstract or efficient strategies. On abstract problems children may be more likely to exhibit the most efficient or abstract strategy which they are capable of using.

Thus, the finding that young children can solve verbal and abstract problems equally well may suggest that initial instruction on addition and subtraction could be based on problems of one type as well as the other, the strategies children use in these two contexts suggest that verbal problems should be included in initial instruction.

Kindergartners' and First-graders' Strategies

Results of the study suggest that many kindergarten and first-grade children are quite capable problem solvers. Most kindergartners made reasonable attempts to solve at least some of the problems, indicating that children at this level are capable of solving both verbal and abstract problems (when problems are read to them). Even though no instruction on missing addend problems had occurred, such problems were appropriately solved by nearly half or more of the subjects in each grade. Further evidence for the problem solving capabilities of these young children is provided by the fact that more than one-fifth of the kindergartners and one-fourth of the first-graders used the strategy involving the derivation of a needed number fact from another known fact (Derived Fact) at least once. In a limited sense, these children demonstrated the ability to apply Polya's (1957) heuristic of solving a simpler or related problem.

The existence of the preceding problem solving capabilities suggests that initial instruction on addition and subtraction might be more effective if it were tailored to assess and extend the capabilities that individuals bring to the instructional process. It would be wise for teachers to determine the strategies children are capable of using and then to encourage the development of individuals' problem solving capabilities by using the Derived Fact strategy as a starting point for introducing certain problem solving heuristics to young children.

Not surprisingly, kindergartners' strategies were less abstract than those of first-graders and reflected an even closer relationship to the structure of the problem. However, many kindergartners did not rely solely on concrete representation strategies; they used abstract strategies but simply used them infrequently. Kindergarten children seldom used strategies interchangeably, e.g., used both additive and subtractive strategies on subtraction problems of a given type. Even the first-graders exhibited less use of concrete representation or counting strategies not reflecting problem structure than did the subjects in Carpenter and Moser (1981).

The increased level of abstraction of first-graders' strategies was also accompanied by a decrease in the frequency of certain errors. Although first-graders used the wrong operation and committed about the same number of procedural errors as kindergartners, they less frequently exhibited errors involving identification of the answer set, incorrect modeling, and superficial solutions such as guessing.

The preceding discussion suggests that kindergartners and first-grade children exemplify different levels in the acquisition of addition and subtraction concepts and skills. At an early level children are more likely to incorrectly model, superficially interpret, or fail to attempt problems; and their correct interpretations are often quite literal, closely mirroring the structure of the problem. Later, children begin to abstract the essential elements of the problem without a visible step-by-step re-creation of the problem. At this level children no longer exhibit incorrect modeling of the problem data, but they may be unable to correctly interpret some problems (use the wrong operation) and still may commit frequent procedural errors such as miscounting or forgetting the data in the problem. At an even later level, beyond that of the first-graders in this study, children may exhibit more flexibility in their strategies; they may recognize the equivalence of various strategies, choose among them, and use them interchangeably for various problems. Children's errors at this level become primarily procedural, seldom involving use of the wrong operation or incorrect representation or identification of the answer.

In order for children to progress optimally through these levels it is necessary for teachers, at the minimum, to be aware of where students fall on this continuum. This involves assessing students' capabilities and using their errors to diagnose misconceptions. The clusterings of subjects in both grades indicate a great deal of variability among subjects in these grades; this is further evidence that instruction is not simply a matter of teaching children the "one way" to do addition and subtraction problems. More attention should be given to determining the processes children use and building further instruction upon what is known about the individual's capabilities.

Limitations of the Study

The selection of subjects from intact classes from two available schools limits the extent to which the findings of the study generalize to other samples of kindergarten and first-grade children. Although the subjects in the study were not atypical youngsters, one cannot assume that the performance of children with different home experiences and different socioeconomic backgrounds would necessarily be the same.

No data were collected on any subject variables other than age. This served to limit interpretation of the results of the study, particularly the clusterings of subjects. It was not possible to relate the solution strategies typically used by clusters of subjects or the types of problems they could solve to other variables such as memory or cognitive processing capacity, socioeconomic status, developmental level, or achievement.

No attempt was made to control the instructional backgrounds of the subjects, so it is possible that children with other instructional experiences could exhibit different performance. This especially might be true if children received more instruction on verbal problems than did the first-graders in the present study. No direct observation of classes was done to corroborate the teachers' accounts of prior instructional experiences. Hence, the possibility exists that certain topics related to verbal and/or abstract addition or subtraction problems were introduced or stressed by being presented to individuals without later recollection by the teacher.

The tasks used in the study limited the results in several ways. First, the only problem types used were those verbal and abstract problems that previous research had shown to be the least difficult for children in these grades. The finding of no differences in children's ability to solve corresponding verbal and abstract problems in this study does not preclude the existence of differences in children's ability to solve verbal and abstract addition and subtraction problems of other types. Likewise, a comparison of the strategies children use on other verbal and abstract problem types may yield differences other than those found in the present study.

Secondly, the use of different modes of presentation (oral and written) for abstract problems limits the comparison of performance on these problems across the two grades. Also, the abstract problems presented orally to the kindergarten subjects may not have been perceived as being substantially different from the verbal problems read to these subjects. Reading the abstract problem to the subject may have transformed it to a type of pseudo-verbal problem. This may have served to suppress any differences between performance on abstract and verbal problems at the kindergarten level.

A third limitation resulting from the tasks used in the study derived from the wording of the verbal problems. Inclusion of cue words such as "altogether," "left," and "put with" may have influenced the strategies subjects used on the verbal problems. Different wordings may be less suggestive of some of the concrete representation strategies that were used.

The procedures used in the interviews generated several limitations. Foremost among these were the subjective decisions made by the interviewers when coding subjects' responses. The interviewers occasionally encountered subjects who purported to recall number facts but generated incorrect facts. These responses were distinguished from guessing; however, such distinctions can easily be questioned. Likewise, kindergartners' use of number facts on problems with sums less than six was distinguished from counting on or counting back on the basis of the subject's verbalizations. Again, since counting which involves so few counts may be difficult to distinguish from recall of facts, the accuracy of the coding of some of the kindergartners' responses on problems with sums less than six can be questioned.

The decision to make manipulatives available for all kindergarten subjects precluded any between-grade comparison of strategies used when no manipulatives are available. Other studies (e.g., Moser, Note 5) have shown that the availability of manipulatives influences the strategies children use to solve addition and subtraction problems. It is likely that kindergartners who had no manipulatives available would exhibit somewhat different strategies on the problems used in this study. The strategies used by the kindergartners in the present study are not necessarily those that would have been used by the subjects if manipulatives had not been available.

Use of standard sets of tasks for the interviews limited the extent to which they were able to assess children's capabilities to solve various types of problems. For example, if a subject guessed or appeared not to know how to solve a certain problem, this one attempt was accepted as an assessment of the child's performance on such a problem. The interviewer was not free to pursue several examples of one problem type or to reword or revise problems during the interview. This made it impossible to gather the variety of data which can be garnered from a true clinical interview.

Implications for Future Research

As is often the case, the present study raises as many questions as it has answered. The limitations of the study suggest a number of extensions, and the results of the study lead to additional researchable questions. These potential research areas fall into the three categories which follow.

Extensions of the Problem Domain

The study revealed no differences (as measured by correctness or use of an appropriate strategy) in children's ability to solve verbal and abstract problems of the form a+b=__, a-b=__, and a+__=c; future research should investigate differences on corresponding abstract and verbal problems based on a-__=c, __+b=c, and __-b=c. Verbal problems might be worded both to minimize the influence of cue words or to make maximum use of such wordings to heighten the differences between verbal problems and their abstract counterparts.

By administering problems from the present study and the extended domain discussed previously with children in kindergarten through grade three, one could get a more complete picture of children's performance on verbal and abstract problems. Data on problems drawn from a variety of number size levels administered to subjects in second and third grade would provide information from children who have received substantial instruction on the addition and subtraction algorithms and would highlight the effects of such instruction on both verbal and abstract problems. One might also use a variety of problem types to determine, by means of clinical interviews, whether older children's counting strategies are based on problem structure or the efficiency of alternative counting procedures as suggested by Woods et al. (1975). A further interesting extension would be to include two-step addition and subtraction problems since these typically are the most difficult for children in later grades.

Assessment of Individual Differences

The results of the cluster analyses demonstrate the viability of differentiating among individuals according to their solution strategies. Particularly at the first-grade level, identifying children by the strategies they use may yield more information than classifying children according to the types of problems they can solve.

The present study also demonstrates the viability of using verbal and abstract problems within the partially standardized clinical interview procedure for children at the kindergarten level. A reasonable next step would be to use more intensive clinical interviews with kindergarten and first-grade children to explore in depth the difficulties children encounter on certain problems, to assess the effectiveness of brief but intensive individual instruction for diminishing those difficulties, to further examine young children's use of problem solving heuristics, to attempt to determine how children choose among alternatives within their repertoire of strategies, and to investigate the extent to which children mentally manipulate or transform certain problems presented in the abstract number sentence context prior to solving them.

An in-depth assessment of children who have exceptional difficulty with certain addition and subtraction problems or whose language difficulties contribute to their inability to solve verbal problems might yield further suggestions for instruction for such children. In a more genuine clinical interview it might be possible for the interviewer to gain insight into the reasons why children who are capable of using concrete representation, counting and mental strategies use one strategy for a certain problem and a different one for a similar problem. In the past researchers have seldom been successful in identifying children's motives for such choices; perhaps by allowing the interviewer to demonstrate the equivalence of different modeling or counting procedures or to create conflict among different solution procedures, the clinical interview might shed light on some of the less visible or verbalizable aspects of children's solution processes.

Interview data can also provide information about how children interpret and manipulate problems in number sentence form. Young children often solve verbal problems without translating problems to number sentences. It is possible that children occasionally think of their modeling of abstract problems in terms of some form of a verbal problem. Such a relationship between abstract and verbal problems can have important implications for the sequencing of instruction entailing problems in verbal and abstract contexts. A more in-depth interview might also provide insights into potential subject variables, for example, information processing capacity, that might account for differing levels of children's acquisition of addition and subtraction concepts and skills and the range of individual differences encountered among the subjects in the present study.

Instruction on Addition and Subtraction

The ultimate goal of status studies such as this one is to provide information which can yield direction for adapting instruction to the capabilities and needs of the learners. While the present study can in no way prescribe optimal initial instruction on addition and subtraction, it can suggest some directions for research relating to instruction. Foremost among these is research aimed at comparing various emphases and/or sequences of introduction of verbal and abstract problems in intial instruction relating to addition and subtraction. The clusterings of subjects, especially at the kindergarten level, indicated that children are differentially capable of solving various types of addition and subtraction problems. Research is needed to determine whether these children also may profit differentially from instruction which is sequenced in a given way.

Another more general area which may be fruitful for research is the assessment of young children's use of problem solving heuristics. The present study indicates that some young children are capable of using certain problem solving heuristics and that they somehow are able to choose strategies from among a variety of alternatives. In order for the mathematics curriculum to incorporate instruction on such aspects of problem solving, much more research needs to be done to identify those mathematical problem solving processes that develop early as well as when and how instruction can enhance that development.

Conclusion

Three principal conclusions can be drawn from this study. First, at the kindergarten and first-grade levels it is not the case that verbal or word problems are more difficult than corresponding abstract or symbolic problems. Children in both grades performed equally well on problems in the two contexts, indicating that verbal problems are an appropriate adjunct to abstract problems for initial instruction on addition and subtraction.

A second conclusion is that many young children use a variety of strategies to solve addition and subtraction problems even though strategies are often closely tied to problem structure. They also use somewhat different strategies on verbal and abstract problems. These differences, namely, more frequent use of concrete representation strategies on verbal problems and more guessing, counting, and mental strategies on abstract problems, suggest that verbal problems may be the most appropriate ones for initial instruction.

A third conclusion is that although children in grades K and 1 appear to begin to progress toward more abstract strategies, there is substantial variabilty among children both in terms of the strategies they typically use and the types of problems they are capable of solving. In addition to the variability among subjects in each grade level, the variety of strategies used by individuals suggests that, in some cases, comparable variability exists within individuals. Thus, it is important for instruction to be designed to take into account such individual differences.

Brownell (1941) argued that experiences with the number combinations must be "well-chosen and wisely directed" (p. 44). By documenting young children's solution processes for verbal and abstract addition and subtraction problems the present study has contributed data from which choices concerning appropriate initial instructional experiences relating to addition and subtraction can be better determined.

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APPENDIX A

VERBAL PROBLEM STEMS

Verbal Problem Stems

Join

Judy had __ stamps. Her mother gave her __ more stamps. How many stamps did Judy have altogether? **

Wally had __ pennies. His father gave him __ more pennies. How many pennies did Wally have altogether? *

Combine

Fred saw __ tigers. He also saw __ elephants. How many animals did Fred see altogether? **

Sara has __ sugar donuts. She also has __ plain donuts. How many donuts does Sara have altogether? *

Separate

Joan had __ apples. She gave __ to Leroy. How many apples did Joan have left? **

Mike had __ kites. He gave __ kites to Kathy. How many kites did Mike have left? **

Tim had __ stars. He gave __ stars to Martha. How many stars did Tim have left? *

Ann had __ balloons. She gave __ balloons to Willie. How many balloons did Ann have left? *

Join/Change Unknown

Joe has __ books. How many more books does he have to put with them so he has __ books altogether? **

Susan has __ cookies. How many more cookies does she have to put with them so she has __ cookies altogether? **

Kathy has __ pencils. How many more pencils does she have to put with them so she has __ pencils altogether? *

John has __ cats. How many more cats does he have to put with them so he has __ cats altogether? *

* - Used for grade K, small number problems and grade 1, larger number problems

** - Used for grade K, larger number problems and grade 1, small number problems

APPENDIX B

NUMBER TRIPLE ORDERS

Number Triple Orders

Kindergarten (small numbers)

	Problem *					
Order	a+b=__	a-b=__	a+__=c	a+b=__	a-b=__	a+__=c
1	1,4,5	2,3,5	1,3,4	2,3,5	1,3,4	1,4,5
2	2,3,5	1,3,4	1,4,5	1,3,4	1,4,5	2,3,5
3	1,3,4	1,4,5	2,3,5	1,4,5	2,3,5	1,3,4
4	1,3,4	2,3,5	1,4,5	2,3,5	1,4,5	1,3,4
5	2,3,5	1,4,5	1,3,4	1,4,5	1,3,4	2,3,5
6	1,4,5	1,3,4	2,3,5	1,3,4	2,3,5	1,4,5

* These orders were assigned to problem types rather than to a sequence of problems; this was done to prevent the same triple from appearing twice for a given problem type.

Kindergarten (larger numbers) and First-grade (small numbers)

	Order of Problem in Sequence of Tasks					
Order	1	2	3	4	5	6
1	3,6,9	2,5,7	2,7,9	2,4,6	3,5,8	2,6,8
2	2,6,8	3,5,8	2,4,6	3,6,9	2,7,9	2,5,7
3	2,5,7	2,6,8	3,6,9	3,5,8	2,4,6	2,7,9
4	3,5,8	2,4,6	2,6,8	2,7,9	2,5,7	3,6,9
5	2,4,6	2,7,9	2,5,7	2,6,8	3,6,9	3,5,8
6	2,7,9	3,6,9	3,5,8	2,5,7	2,6,8	2,4,6

First-grade (larger numbers)

Order	Order of Problem in Sequence of Tasks 1	2	3	4	5	6
1	6,9,15	4,9,13	3,8,11	4,7,11	4,8,12	5,9,14
2	5,9,14	4,8,12	4,7,11	6,9,15	3,8,11	4,9,13
3	4,9,13	5,9,14	6,9,15	4,8,12	4,7,11	3,8,11
4	4,8,12	4,7,11	5,9,14	3,8,11	4,9,13	6,9,15
5	4,7,11	3,8,11	4,9,13	5,9,14	6,9,15	4,8,12
6	3,8,11	6,9,15	4,8,12	4,9,13	5,9,14	4,7,11

APPENDIX C

TASK ORDERS

Task Orders

Interview Task

Task Order	1	2	3	4	5	6
1	A2*	S3	S2	S4	A1	S1
2	S2	A1	S3	S1	S4	A2
3	A1	S1	S4	A2	S2	S3
4	S1	A2	S4	A1	S3	S2
5	A2	S4	S1	S3	S2	A1
6	A1	S2	S3	S1	A2	S4

* Tasks:

A1 - Verbal: Join
Abstract: a+b=__

A2 - Verbal: Combine
Abstract: a+b=__

S1 - Verbal: Separate (small difference)
Abstract: a-b=__ (small difference)

S2 - Verbal: Separate (large difference)
Abstract: a-b=__ (large difference)

S3 - Verbal: Join/Change Unknown (small difference)
Abstract: a+__=c (small difference)

S4 - Verbal: Join/Change Unknown (large difference)
Abstract: a+__=c (large difference)

APPENDIX D

SAMPLE INTERVIEWS

Sample Interviews

Problems Administered to Kindergarten Subject #11

First Interview -- Abstract Problems

Problems with small numbers (sums less than 6)

1. "Five take away three is how many?"
2. "One and three are how many?"
3. "One and how many are five?"
4. "Two and three are how many?"
5. "Three and how many are four?"
6. "Five take away one is how many?"

Problems with larger numbers (sums 6 through 9)

1. "Three and six are how many?"
2. "Five and how many are seven?"
3. "Nine take away two is how many?"
4. "Two and how many are six?"
5. "Three and five are how many?"
6. "Eight take away six is how many?"

Second Interview -- Verbal Problems

Problems with small numbers (sums less than 6)

1. "Sara has one sugar donut. She also has four plain donuts. How many donuts does Sara have altogether?"

2. "John has two cats. How many more cats does he have to put with them so he has five cats altogether?"

3. "Tim had four stars. He gave three stars to Martha. How many stars did Tim have left?"

4. "Kathy has four pencils. How many more pencils does she have to put with them so she has five pencils

altogether?"

5. "Ann had five balloons. She gave two balloons to Willie. How many balloons did Ann have left?"

6. "Wally had one penny. His father gave him three more pennies. How many pennies did Wally have altogether?"

Problems with larger numbers (sums 6 through 9)

1. "Judy had two stamps. Her mother gave her five more stamps. How many stamps did Judy have altogether?"

2. "Mike had eight kites. He gave two kites to Kathy. How many kites did Mike have left?"

3. "Joe has six books. How many more books does he have to put with them so he has nine books altogether?"

4. "Joan had eight apples. She gave five apples to Leroy. How many apples did Joan have left?"

5. "Fred saw two tigers. He also saw four elephants. How many animals did Fred see altogether?"

6. "Susan has two cookies. How many more cookies does she have to put with them so she has nine cookies altogether?"

<u>Problems Administered to First-grade Subject #2</u>

First Interview -- Verbal Problems

Problems with small numbers (sums 6 through 9)

1. "Mike had eight kites. He gave two kites to Kathy. How many kites did Mike have left?"

2. "Judy had three stamps. Her mother gave her five more stamps. How many stamps did Judy have altogether?"

3. "Joe has four books. How many more books does he have to put with them so he has six books altogether?"

4. "Joan had nine apples. She gave six apples to Leroy. How many apples did Joan have left?"

5. "Susan has two cookies. How many more cookies

does she have to put with them so she has nine cookies altogether?"

6. "Fred saw two tigers. He also saw five elephants. How many animals did Fred see altogether?"

Problems with larger numbers (sums 11 through 15)

1. "Sara has four sugar donuts. She also has eight plain donuts. How many donuts does Sara have altogether?"

2. "John has four cats. How many more cats does he have to put with them so he has eleven cats altogether?"

3. "Tim had fourteen stars. He gave nine stars to Martha. How many stars did Tim have left?"

4. "Kathy has eight pencils. How many more pencils does she have to put with them so she has eleven pencils altogether?"

5. "Ann had thirteen balloons. She gave four balloons to Willie. How many balloons did Ann have left?"

6. "Wally had six pennies. His father gave him nine more pennies. How many pennies did Wally have altogether?"

Second Interview -- Abstract Problems

Problems with small numbers (sums 6 through 9)

1. 2 + 4 = __

2. 9 - 7 = __

3. 2 + __ = 7

4. 2 + 6 = __

5. 9 - 3 = __

6. 5 + __ = 8

Problems with larger numbers (sums 11 through 15)

1. 11 - 8 = __

2. 6 + 9 = __

3. 4 + __ = 12

4. 4 + 9 = __

5. 9 + __ = 14

6. 11 - 4 = __

APPENDIX E

INTERVIEW PROTOCOLS

Interview Protocols

Warm Up Tasks

Here are some objects. I'm going to sort the objects into two piles. (THREE YELLOW TRIANGLES ARE PUT INTO ONE PILE AND THREE BLUE RECTANGLES INTO ANOTHER.)

This piece (A YELLOW RECTANGLE) is yours. You decide which pile you'd like to put it into. (CHILD PLACES OBJECT INTO PILE.)

Very good. How did you decide to put it over there? (CHILD DISCUSSES WHY ONE PILE WAS CHOSEN.)

(A SET OF SEVEN CUBES IS PLACED ON THE TABLE.) Here is a pile of blocks. How many blocks are in this pile? (CHILD ANSWERS.)

(A SET OF FOUR FINGERS IS HELD UP.) How many fingers am I holding up? (CHILD ANSWERS.) (IF AN INCORRECT ANSWER OR ANSWERS ARE GIVEN, A SIMPLER PROBLEM SUCH AS ENUMERATING A SET OF 2 FINGERS SHOULD BE GIVEN AND THEN A FURTHER ATTEMPT MADE TO GET THE CHILD TO RESPOND CORRECTLY TO THE PROBLEM ANSWERED INCORRECTLY.)

(TWENTY CUBES ARE PLACED ON THE TABLE.) Make a pile of six blocks for yourself. (CHILD FORMS A SET OF CUBES.)

Can you hold up four fingers? (CHILD HOLDS UP FINGERS.) (IF AN INCORRECT ANSWER OR ANSWERS ARE GIVEN, A SIMPLER PROBLEM SHOULD AGAIN BE GIVEN AND THEN A FURTHER ATTEMPT MADE TO GET THE CHILD TO RESPOND CORRECTLY TO THE PROBLEM ANSWERED INCORRECTLY.)

Addition and Subtraction Tasks

I'm going to read you some number stories (number puzzles). Each story has a question. Sometimes I may ask you how you figured out your answer. Since I can't remember everything you say, I'll be writing some things on this paper. Here are some cubes that you can use to help you answer the questions. (PUT CUBES ON TABLE.) You may use the cubes or your fingers or anything else that you think will help you answer the questions. Here's the first story. (READ THE PROBLEM. IF THE CHILD ASKS FOR IT TO BE REPEATED, REREAD THE PROBLEM AS OFTEN AS REQUESTED. IF THE CHILD LOOKS PUZZLED, SUGGEST

REREADING.)

(For first-grade abstract problems use this modification.)

(PUT THE CARD WITH THE PROBLEM IN FRONT OF THE CHILD.) Can you read this for me? (Other versions -- How do you read this one? or What does this say?) (IF THE CHILD DOES NOT READ THE PROBLEM CORRECTLY CODE THIS, NOTING THE MISTAKE. CORRECTLY READ THE PROBLEM FOR THE CHILD.)

(ONCE THE PROBLEM HAS BEEN READ CORRECTLY ASK THE CHILD TO SOLVE IT.) What number should be in the box? (Other versions -- What number should this (POINT TO BOX) be? or Can you tell me what number goes here?)

(CODE THE CHILD'S RESPONSES. DO NOT PROVIDE ASSISTANCE. IF THE CHILD ASKS FOR HELP, RESPOND WITH A NEUTRAL STATEMENT SUCH AS: See if you can figure this one out.) (QUESTION THE CHILD AS NECESSARY TO CLARIFY AMBIGUOUS STATEMENTS OR ACTIONS.)

(IF CUBES WERE USED PUT THEM BACK INTO THE PILE.) Here's the next story. (REPEAT THIS PROCEDURE FOR THE OTHER FIVE SMALL NUMBER PROBLEMS.)

(PAUSE BEFORE READING THE PROBLEMS WITH LARGER NUMBERS.) Here are some stories. Remember, if you wish, you can use the cubes or your fingers or anything else to help you answer the questions.

(REPEAT THE PROCEDURE USED FOR THE SIX SMALL NUMBER PROBLEMS WHEN PRESENTING THE LARGER NUMBER PROBLEMS.)

<u>Debriefing Task</u>

(PUT THE SET OF GEOMETRIC PIECES ON THE TABLE.) Here are some pieces. Put them into two piles any way you'd like. (DISCUSS THE CHILD'S SOLUTION.)

APPENDIX F

CODING SHEET

Coding Sheet

ID NUMBER			AGE	
S	TEACHER	PUPIL	YR	MON

NAME

SEX: M F

ADMINISTRATION 1 2 3 4 5 6

GENERAL TASK CODE a+ b+ b c+ c

	MODEL		STRATEGY	EXPLAIN	ERROR
TASK 1	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S
TASK 2	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S
TASK 3	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S
TASK 4	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S
TASK 5	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S
TASK 6	C F N	UNCODABLE CORRECT Y N	ADDITION: CS CL CA; SUBTRACTION: SEPARATE F T, MATCH, ADD ON; COUNT DF DT UG	HEURISTIC, # FACT, GUESS	MISCOUNT M, GIVEN # G, FORGETS F, OPERATION O, SENTENCE S

APPENDIX G

COMPARISON OF DATA FROM TWO SCHOOLS

Comparison of Data From Two Schools

Percentage of Correct Answers

Problem Type (aggregated across both number sizes and both contexts)	School	Correct	Incorrect	No Attempt
a+b=__	A	73	25	2
	B	64	34	2
a+b=__	A	75	23	1
	B	64	34	2
a-b=__	A	57	41	2
(small difference)	B	57	38	5
a-b=__	A	63	37	0
(large difference)	B	61	35	5
a+__=c	A	57	41	2
(small difference)	B	53	43	3
a+__=c	A	46	51	3
(large difference)	B	46	49	5

Percentage of Use of Most Frequently Used Strategies by Subjects from Two Schools

Problem Type (aggregated across both number sizes and both contexts)	School	CA	S	CS	CL	#F	GU	G#	OP	No Attempt
a+b=__	A	34	2	11	14	22	7	3	1	2
	B	31	7	15	10	19	13	8	1	2
a+b=__	A	36	3	12	15	19	7	2	1	1
	B	29	11	6	8	22	13	7	1	2

Strategy *

Problem Type	School	F	DF	DT	DR	#F	GU	G#	OP	No Attempt
a-b=__ (small difference)	A	50	7	0	1	13	11	3	6	2
	B	42	5	4	4	17	15	6	1	5
a-b=__ (large difference)	A	42	13	0	0	20	11	2	5	0
	B	36	10	1	0	27	13	5	1	5

Problem Type	School	F	AO	AG	UG	#F	GU	G#	OP	No Attempt
a+__=c (small difference)	A	2	27	3	18	16	13	3	13	2
	B	2	21	2	15	25	16	5	5	3
a+__=c (large difference)	A	3	30	3	12	10	11	3	20	3
	B	3	28	5	9	16	15	9	4	5

* Strategies:

CA - Counting All
CS - Counting On From Smaller
F - Separate From
AG - Add On/Given Number
DT - Count Down To
DR - Derived Fact
GU - Guess
OP - Wrong Operation
S - Subitize
CL - Counting On From Larger
AO - Adding On
DF - Count Down From
UG - Count Up From Given
#F - Number Fact
G# - Given Number

APPENDIX H

FREQUENCY OF USE OF APPROPRIATE STRATEGIES ON SIX PROBLEM TYPES

Frequency of Use of Appropriate Strategies On Six Problem Types

Problem Type

Subject Number	Verbal (a+b=__)	Verbal (a-b=__)	Verbal (a+__=c)	Abstract (a+b=__)	Abstract (a-b=__)	Abstract (a+__=c)
<u>Grade K</u>						
1	4	4	4	3	3	4
2	4	4	4	4	4	4
3	2	2	2	0	0	0
4	4	4	4	3	3	3
5	4	4	4	4	3	0
6	4	4	2	4	3	4
7	1	0	0	0	0	0
8	3	4	3	1	4	2
9	4	4	3	4	2	4
10	4	0	0	4	0	0
11	4	3	3	2	4	1
12	1	0	0	1	0	0
13	3	4	0	4	3	0
14	4	4	2	4	2	1
15	4	4	3	4	4	4
16	3	3	4	4	1	3
17	4	1	3	4	0	2
18	4	4	4	4	4	3
19	4	3	1	4	1	0
20	4	3	2	4	0	1
21	4	4	3	4	4	4
22	4	1	0	4	2	0
23	1	4	0	0	0	0
24	4	4	4	4	4	3
25	4	3	2	3	4	3
26	0	0	0	3	0	0
27	0	0	0	2	1	3
28	4	4	4	4	4	3
29	4	1	2	4	2	3
30	4	4	3	4	4	0
31	3	3	2	3	4	1
32	4	4	4	4	4	4
33	3	0	3	4	4	2
34	3	3	1	4	2	0
35	1	1	0	0	0	0
36	4	4	4	4	4	4

Problem Type

Subject Number	Verbal (a+b=__)	Verbal (a-b=__)	Verbal (a+__=c)	Abstract (a+b=__)	Abstract (a-b=__)	Abstract (a+__=c)
Grade K						
37	4	4	3	4	4	3
38	1	0	0	0	0	0
39	3	4	4	4	3	1
40	3	4	4	4	4	4
41	4	2	3	4	1	1
42	1	1	1	1	1	0
43	4	4	4	4	4	4
44	1	0	0	2	0	0
45	4	3	4	4	4	4
46	0	0	0	0	0	0
47	4	4	3	4	4	4
48	2	2	2	1	1	0
49	4	3	3	3	2	2
50	4	4	2	4	4	3
Grade 1						
1	4	4	4	4	4	4
2	4	4	4	4	4	4
3	4	4	4	4	4	4
4	4	1	0	4	4	0
5	2	4	2	3	4	1
6	4	2	2	2	2	0
7	4	4	3	4	4	3
8	4	4	3	4	4	4
9	4	3	4	4	4	0
10	4	4	4	4	4	4
11	3	2	2	2	3	4
12	4	4	4	4	4	4
13	4	4	4	4	4	3
14	4	0	0	4	0	0
15	4	4	2	4	4	4
16	4	4	4	4	3	3
17	4	0	3	4	4	3
18	2	4	1	3	4	2
19	4	4	4	4	4	4
20	4	4	4	4	4	2
21	4	4	4	4	4	4
22	4	4	3	4	4	4

Problem Type

Subject Number	Verbal (a+b=__)	Verbal (a-b=__)	Verbal (a+__=c)	Abstract (a+b=__)	Abstract (a-b=__)	Abstract (a+__=c)
<u>Grade 1</u>						
23	4	3	4	4	2	3
24	4	4	4	3	3	3
25	2	4	1	4	4	1
26	4	3	4	4	3	4
27	4	4	4	4	2	4
28	4	4	4	4	3	4
29	4	3	3	4	4	0
30	4	4	4	4	3	4
31	4	4	2	3	4	4
32	0	0	0	3	3	0
33	4	4	1	3	4	2
34	4	3	2	4	4	2
35	4	3	3	4	4	4
36	4	4	2	4	4	4
37	4	4	3	4	3	3
38	4	3	4	3	4	4
39	2	0	0	4	2	0
40	3	4	2	4	3	2
41	3	4	3	4	4	4
42	1	4	0	2	2	2
43	4	4	2	0	0	1
44	0	2	1	3	4	2
45	2	4	3	4	2	4
46	4	4	4	4	4	4
47	4	4	4	4	4	4
48	3	3	2	2	3	2
49	3	4	3	4	4	2
50	4	4	4	4	4	4
51	3	4	4	4	4	3
52	4	4	4	4	4	4
53	4	4	4	4	4	4
54	4	4	4	4	4	4

APPENDIX I

FREQUENCY OF USE OF STRATEGIES

Frequency of Use of Strategies

Strategy Abbreviations:

CA - Counting All

S - Subitize

CS - Counting On From Smaller Addend

CL - Counting On From Larger Addend

F - Separate From

T - Separate To

AO - Adding On

AG - Add On/Given Number

DF - Counting Down From

DT - Counting Down To

UG - Counting Up From Given

DR - Derived Fact

#F - Number Fact

GU - Guess

IF - Inappropriate Fact

G# - Given Number

OP - Wrong Operation

MBS - Models Both Sets

ACU - All Cubes Used

XU - Uncodable

No - No Attempt

Strategy

Subject Grade K	CA	S	CS	CL	F	T	AO	AG	DF	DT	UG	DR	#F	GU	IF	G#	OP	MBS	ACU	U	NO
01	1	0	3	1	1	0	0	0	4	0	7	0	5	1	0	1	0	0	0	0	0
02	4	0	0	2	8	0	8	0	0	0	0	0	2	0	0	0	0	0	0	0	0
03	0	0	0	0	0	0	0	0	0	0	0	0	4	12	0	0	0	0	0	4	4
04	7	0	0	0	8	0	6	0	0	0	0	0	0	0	1	1	0	1	0	0	0
05	1	0	0	1	5	0	1	0	0	0	2	0	9	0	0	4	1	0	0	0	0
06	6	2	0	0	6	0	5	0	0	0	0	0	2	0	0	0	3	0	0	0	0
07	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	0	0	0	0	1	0
08	4	0	0	0	8	0	5	3	0	0	0	0	0	0	0	1	3	0	0	0	0
09	6	0	0	2	4	0	3	0	1	0	4	0	1	0	1	0	1	1	0	0	0
10	8	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	11	0	1	0	1
11	4	0	0	0	5	0	3	1	1	0	0	1	3	2	0	2	1	1	0	0	0
12	1	0	0	0	0	0	0	0	0	0	0	0	0	18	0	1	3	0	0	1	0
13	7	0	0	0	5	0	0	1	0	0	0	0	2	3	0	0	6	0	0	0	0
14	8	0	0	0	6	0	3	2	0	0	0	0	0	0	0	1	3	1	0	0	0
15	7	0	1	0	8	0	7	0	0	0	0	0	0	0	0	1	0	0	0	0	0
16	5	0	2	0	3	0	4	0	0	0	2	0	1	2	0	1	2	0	0	2	0
17	0	0	3	3	0	0	0	0	0	0	3	1	1	5	0	3	2	0	0	3	0
18	2	0	0	2	4	0	3	0	0	0	0	1	11	0	1	0	0	0	0	0	0
19	1	0	1	1	2	0	0	0	0	0	0	0	7	0	0	2	6	0	0	2	2
20	6	1	0	0	3	0	3	1	0	0	0	0	1	1	0	3	2	0	0	3	0
21	6	2	0	0	9	0	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0
22	8	0	0	0	3	0	0	1	0	0	0	0	0	0	0	0	11	0	0	0	1
23	0	0	0	0	0	0	0	0	0	0	0	0	5	13	0	2	4	0	0	0	0
24	7	0	0	0	8	0	7	0	0	0	0	0	1	0	0	1	0	0	0	0	0
25	4	2	1	0	3	0	0	0	2	0	3	0	4	2	0	1	0	0	0	0	2
26	2	0	0	0	0	0	0	1	0	0	0	0	1	20	0	0	0	0	0	0	0
27	1	0	0	1	1	0	2	0	0	0	0	1	0	12	0	3	1	1	1	0	0
28	8	0	0	0	8	0	5	1	0	0	0	0	2	0	0	0	0	0	0	0	0
29	8	0	0	0	1	0	5	0	0	0	0	0	2	3	1	1	2	1	0	0	0
30	4	0	0	0	7	0	2	1	0	0	0	0	6	2	0	0	2	0	0	0	0
31	3	0	0	0	4	0	1	0	0	0	0	1	4	5	0	0	2	0	0	3	1
32	5	0	0	0	6	0	5	0	0	0	1	1	5	0	0	0	0	0	0	1	0
33	5	2	0	0	3	0	5	0	0	0	0	0	1	5	0	1	0	0	0	0	2
34	3	3	1	0	4	0	0	4	0	0	0	0	2	1	0	1	0	1	0	0	4
35	1	0	0	0	1	0	0	2	0	0	0	0	0	11	0	5	1	0	0	0	3
36	0	0	0	2	0	0	0	0	1	0	2	2	17	0	0	0	0	0	0	0	0
37	6	2	0	0	8	0	5	1	0	0	0	0	1	0	0	0	1	0	0	0	0
38	1	0	0	0	0	0	0	0	0	0	0	0	0	7	0	15	0	0	0	0	1
39	7	0	0	0	5	0	4	0	0	0	0	0	3	0	0	1	2	2	0	0	0
40	1	0	0	0	4	0	2	0	1	0	0	1	14	0	0	0	1	0	0	0	0

Strategy

Subject	CA	S	CS	CL	F	T	AO	AG	DF	DT	UG	DR	#F	GU	IF	G#	OP	MBS	ACU	U	NO
Grade K																					
41	7	1	0	0	4	0	3	0	0	0	0	0	[illegible]	0	1	4	0	2	0	1	1
42	0	0	0	0	0	0	0	0	0	0	0	0	0	13	0	4	0	0	0	7	0
43	5	0	2	1	7	0	8	0	0	0	0	0	1	0	0	0	0	0	0	0	0
44	0	0	1	0	0	0	0	0	0	0	0	0	0	14	0	3	2	0	0	2	2
45	4	2	0	0	7	0	6	0	0	0	0	1	3	1	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	13	0	11	0	0	0	0	0
47	4	2	0	0	9	0	2	1	0	0	1	2	3	0	0	0	0	0	0	0	0
48	0	0	0	1	0	0	0	0	2	0	2	0	3	10	0	5	0	0	0	1	0
49	4	0	0	0	3	0	3	0	0	0	0	1	6	1	0	4	2	0	0	0	0
50	2	2	0	1	5	0	3	0	0	0	0	0	8	0	0	0	2	0	0	1	0

Grade 1

Subject	CA	S	CS	CL	F	T	AO	AG	DF	DT	UG	DR	#F	GU	IF	G#	OP	MBS	ACU	U	NO
01	3	0	0	3	6	0	3	0	0	0	2	2	5	0	0	0	0	0	0	0	0
02	0	0	3	1	0	0	0	0	4	0	6	1	9	0	0	0	0	0	0	0	0
03	4	0	0	2	8	1	4	0	0	0	1	0	4	0	0	0	0	0	0	0	0
04	1	1	0	4	4	1	0	0	0	0	0	0	2	1	0	1	9	0	0	0	0
05	1	0	1	1	4	0	0	0	0	0	0	0	9	0	0	1	1	0	0	1	5
06	3	0	0	1	1	0	0	0	0	0	0	0	7	7	0	0	3	0	0	0	2
07	5	0	0	0	8	0	2	0	0	0	1	0	6	0	0	0	0	0	0	1	1
08	4	0	1	2	8	0	6	0	0	0	0	0	2	0	0	0	1	0	0	0	0
09	0	0	0	5	7	0	2	0	0	0	2	0	3	1	0	0	3	0	0	1	0
10	0	0	0	1	0	0	0	0	0	0	1	1	21	0	0	0	0	0	0	0	0
11	2	0	1	0	2	0	2	0	0	0	2	0	7	3	0	0	0	0	0	0	5
12	6	1	0	0	6	0	8	0	0	0	0	0	3	0	0	0	0	0	0	0	0
13	0	0	0	7	2	0	0	0	7	0	6	0	1	1	0	0	0	0	0	0	0
14	0	0	5	0	2	0	0	0	2	0	0	0	3	1	0	0	11	0	0	0	0
15	1	0	0	5	3	0	3	0	4	0	2	0	4	0	0	0	2	0	0	0	0
16	4	0	1	2	6	0	2	0	0	0	4	0	2	1	0	1	0	0	0	1	0
17	4	0	2	0	1	0	5	0	2	0	1	0	3	0	0	0	4	0	0	2	0
18	0	0	1	3	6	0	1	0	2	0	1	0	1	3	0	5	0	0	0	1	0
19	2	0	0	0	0	0	1	0	0	[illegible]	0	1	20	0	0	0	0	0	0	0	0
20	0	0	0	3	3	0	0	0	2	0	1	0	12	1	0	0	0	0	0	1	1
21	0	1	2	2	3	0	3	0	3	0	4	0	6	0	0	0	0	0	0	0	0
22	3	0	3	1	7	0	5	1	1	0	0	0	3	0	0	0	0	0	0	0	0
23	0	0	3	2	1	0	1	0	3	0	3	1	3	5	0	0	0	0	0	2	0
24	0	0	0	2	2	0	0	0	2	0	2	1	11	0	0	0	0	0	0	2	2
25	0	0	0	0	0	0	0	0	0	0	0	0	15	4	0	0	3	0	0	2	0

Strategy

Subject Grade 1	CA	S	CS	CL	F	T	AO	AG	DF	DT	UG	DR	#F	GU	IF	G#	OP	MBS	ACU	U	NO
26	0	0	6	0	3	0	2	0	3	0	2	1	5	0	0	0	1	0	0	1	0
27	1	0	1	3	5	0	1	0	3	0	5	0	0	0	0	0	2	0	0	1	0
28	6	0	2	0	6	0	8	0	0	0	0	0	1	0	0	0	0	0	0	1	0
29	0	0	3	1	2	0	0	0	5	0	3	1	3	0	0	2	4	0	0	0	0
30	2	0	3	1	4	0	2	0	0	0	3	1	7	1	0	0	0	0	0	0	0
31	0	0	0	7	3	0	0	0	2	0	6	0	3	0	0	0	2	0	0	1	0
32	0	0	3	0	3	0	0	0	0	0	0	0	0	0	0	5	0	1	12	0	0
33	0	2	5	0	6	[illegible]	2	0	2	0	1	0	0	1	0	0	4	0	0	0	1
34	0	1	3	2	7	0	3	0	0	0	0	0	3	0	0	0	3	0	0	0	2
35	2	0	0	2	4	0	1	0	0	0	2	0	11	2	0	0	0	0	0	0	0
36	3	1	0	3	6	0	3	1	0	0	0	1	5	0	0	0	1	0	0	0	0
37	4	1	0	0	7	0	3	0	0	0	0	2	4	1	0	0	2	0	0	0	0
38	4	1	0	0	5	0	2	0	0	0	4	0	6	1	0	0	1	0	0	0	0
39	4	1	1	0	2	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	9
40	0	1	1	3	6	0	3	0	0	0	1	0	3	5	0	0	0	0	0	1	0
41	5	0	0	0	8	0	4	1	0	0	0	1	4	0	0	1	0	0	0	0	0
42	1	0	0	0	5	0	0	0	1	0	0	0	4	6	0	2	0	0	0	0	5
43	4	0	0	0	4	0	2	0	0	0	0	0	1	11	0	0	0	0	2	0	0
44	0	0	1	0	0	0	0	0	2	2	1	2	4	12	0	0	0	0	0	0	0
45	3	2	1	0	1	0	4	0	2	1	3	0	2	2	0	3	0	0	0	0	0
46	1	0	0	3	3	0	1	0	1	0	3	0	13	0	0	0	0	0	0	0	0
47	4	1	0	0	2	0	5	0	1	0	1	0	10	0	0	0	0	0	0	0	0
48	0	0	1	0	1	0	0	0	0	0	0	2	11	3	0	0	0	0	0	0	6
49	1	1	0	4	2	0	0	0	2	0	0	0	9	3	0	0	1	0	0	1	0
50	0	0	4	0	0	0	0	0	1	3	5	0	11	0	0	0	0	0	0	0	0
51	1	2	2	0	5	0	3	0	1	0	1	0	7	0	0	1	1	0	0	0	0
52	0	0	0	2	0	0	0	0	3	1	4	2	12	0	0	0	0	0	0	0	0
53	3	0	0	2	5	0	4	0	0	0	1	0	9	0	0	0	0	0	0	0	0
54	0	2	2	3	2	0	0	0	4	0	6	1	4	0	0	0	0	0	0	0	0

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